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ABSTRACT. In this paper, we present some results about blow up of regular solutions to the homogeneous incompressible Navier-Stokes system, in the case of data in the Sobolev space $\dot{H}^s(\mathbb{R}^3)$, where $\dfrac{1}{2} < s < \dfrac{3}{2}\cdot$ Firstly, we will introduce the notion of minimal blow up Navier-Stokes solutions and show that the set of such solutions is not only nonempty but also compact in a certain sense. Secondly, we will state an uniform blow up rate for minimal Navier-Stokes solutions. The key tool is profile theory as established by P. Gérard [17].

1. INTRODUCTION

We consider the Navier-Stokes system for incompressible fluids evolving in the whole space $\mathbb{R}^3$. Denoting by u the velocity, a vector field in $\mathbb{R}^3$, by p in $\mathbb{R}$ the pressure function, the Cauchy problem for the homogeneous incompressible Navier-Stokes system is given by

$$(1) \qquad \left\{\begin{array}{rcl} \partial_t u + u\cdot\nabla u - \Delta u & = & -\nabla p \\ \operatorname{div} u & = & 0 \\ u_{|t=0} & = & u_0. \end{array}\right.$$

Throughout this paper, we will adopt the useful notation $NS(u_0)$ to mean the maximal solution of the Navier-Stokes system, associated with the initial data u_0.

Definition 1.1. *Let s in $\mathbb{R}$. The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$ is the space of tempered distributions u over $\mathbb{R}^3$, the Fourier transform of which belongs to $L^1_{loc}(\mathbb{R}^3)$ and satisfies*

$$\|u\|_{\dot{H}^s} \stackrel{\text{def}}{=} \left(\int_{\mathbb{R}^3} |\xi|^{2s}|\widehat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}} < \infty.$$

It is known that $\dot{H}^s(\mathbb{R}^3)$ is an Hilbert space if and only if $s < \frac{3}{2}$. We will denote by $(\cdot|\cdot)_{\dot{H}^s(\mathbb{R}^3)}$, the scalar product in $\dot{H}^s(\mathbb{R}^3)$. From now on, for the sake of simplicity, it will be an implicit understanding that all computations will be done in the whole space $\mathbb{R}^3$.

Before stating the results we prove in this paper, we recall two fundamental properties of the incompressible Navier-Stokes system. The first one is the conservation of the L^2 energy. Formally, let us take the L^2 scalar product with the velocity u in the equation. We get

$$(2) \qquad \frac{1}{2}\frac{d}{dt}||u(t)||^2_{L^2} + ||\nabla u(t)||^2_{L^2} = -\int_{R^3}\big(u\cdot\nabla u(t)|u(t)\big)_{L^2} - \int_{R^3}\big(\nabla p(t)|u(u)\big)_{L^2}.$$

Thanks to the divergence free condition, obvious integration by parts implies that, for any vector field a

$$(3) \qquad \big(u\cdot\nabla a|a\big)_{L^2} = 0 = \big(\nabla p|a\big)_{L^2}.$$

This gives

$$(4) \qquad \frac{1}{2}\frac{d}{dt}||u(t)||^2_{L^2} + ||\nabla u(t)||^2_{L^2} = 0.$$

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The second property of the system is the scaling invariance. Let us define the above operator:

$$
\forall \alpha \in \mathbb{R}^+,\ \forall \lambda \in \mathbb{R}^+_*,\ \forall x_0 \in \mathbb{R}^3,\quad \Lambda^{\alpha}_{\lambda,x_0}\, u(t,x) \stackrel{\text{def}}{=} \frac{1}{\lambda^{\alpha}} u\Big(\frac{t}{\lambda^2}, \frac{x-x_0}{\lambda}\Big). \tag{5}
$$
$$
\text{If } \alpha = 1,\ \text{ we note } \Lambda^{1}_{\lambda,x_0} = \Lambda_{\lambda,x_0}.
$$

It is easy to see that if u is smooth solution of Navier-Stokes system on $[0,T]\times\mathbb{R}^3$ with pressure p associated with the initial data u_0, then, for any positive λ, the vector field and the pressure

$$
u_\lambda \stackrel{\text{def}}{=} \Lambda_{\lambda,x_0}\, u \quad \text{and} \quad p_\lambda \stackrel{\text{def}}{=} \Lambda^2_{\lambda,x_0}\, p
$$

is a solution of Navier-Stokes system on the interval $[0,\lambda^2 T]\times\mathbb{R}^3$, associated with the initial data

$$
u_{0,\lambda} = \ \Lambda_{\lambda,x_0}\, u_0.
$$

This leads to the definition of scaling invariant space, which is a key notion to investigate local and global well-posedness issues for Navier-Stokes system.

Definition 1.2. *A Banach space X is said to be scaling invariant, if its norm is invariant under the scaling transformation defined by $u \mapsto u_\lambda$*

$$
||u_\lambda||_X = ||u||_X
$$

The first main result on incompressible Navier-Stokes system is due to J. Leray, who proved in [25] in 1934 that given an initial data in the energy space L^2, the associated NS-solutions, called weak solutions, exist globally in time. The key ingredient of the proof is the L^2-energy conservation (4). Moreover, such solutions are unique in 2-D; but the uniqueness in 3-D is still an open problem. One way to adress this question of unique solvability in 3-D is to demand smoother initial data. In this case, we definitely get a unique solution, but the other side of coin is that the problem is only locally well-posed (and becomes globally well-posed under a scaling invariant smallness assumption on the initial data). J. Leray stated such a theorem of existence of solutions, which he called semi-regular solutions.

Theorem 1.1. *Let an initial data u_0 be a divergence free vector field in L^2 such that ∇u_0 belongs to L^2. Then, there exists a positive time T, and a unique solution $NS(u_0)$ in $\mathcal{C}^0([0,T],\dot{H}^1)\ \cap\ L^2([0,T],\dot{H}^2)$. Moreover, a constant c_1 exists such that if $||u_0||_{L^2}\,||\nabla u_0||_{L^2} \leqslant c_1$, then T can be chosen equal to ∞.*

The reader will have noticed that the quantity $||u_0||_{L^2}\,||\nabla u_0||_{L^2}$ is scaling invariant under the operator Λ_{λ,x_0}. Actually, that is the starting point of many frameworks concerning the global existence in time of solutions under a scaling invariant smallness assumption on the data. The celebrated first one was introduced in 1964, by H. Fujita and T. Kato. These authors stated a similar result as J. Leray, but they demanded less regularity on the data. Indeed, they proved that for any initial data in $\dot{H}^{\frac12}$, there exists a positive time T and there exists a unique solution $NS(u_0)$ belonging to $\mathcal{C}^0([0,T],\dot{H}^{\frac12})\cap L^2([0,T],\dot{H}^{\frac32})$. Moreover, if $\|u_0\|_{\dot{H}^{\frac12}}$ is small enough, then the solution is global in time. This theorem can be proved by a fixed-point argument and the key ingredient of the proof is that the Sobolev space $\dot{H}^{\frac12}$ is invariant under the operator Λ_{λ,x_0}. In other words, the Sobolev space $\dot{H}^{\frac12}$ has exactly the same scaling as Navier-Stokes equation. We refer the reader to [1], [13] or [24] for more details of the proof. But in this paper, we are not interested in the particular kind of space. On the contrary, we work with initial data belonging to homogeneous Sobolev spaces, $\dot{H}^s$ with $\frac{1}{2} < s < \frac{3}{2}$, which means that we are above the natural scaling of the equation. The first thing to do is to provide an existence theorem of Navier-Stokes solutions with data in such Sobolev spaces $\dot{H}^s$. The Cauchy problem is known to be locally well-posed; it can be proved by a fixed-point procedure in an adequate function space (we refer the reader to the book [24], from page 146 to 148, of P-G. Lemarié-Rieusset).

We shall constantly be using the following simplified notations:

$$L^\infty_T(\dot H^s) \stackrel{\text{def}}{=} L^\infty([0,T],\dot H^s) \quad \text{and} \quad L^2_T(\dot H^{s+1}) \stackrel{\text{def}}{=} L^2([0,T],\dot H^{s+1}).$$

Let us define the relevant function space we shall be working with in the sequel:

$$X^s_T \stackrel{\text{def}}{=} L^\infty_T(\dot H^s) \cap L^2_T(\dot H^{s+1}), \quad \text{equipped with} \quad \|u\|^2_{X^s_T} \stackrel{\text{def}}{=} \|u\|^2_{L^\infty_T(\dot H^s)} + \|u\|^2_{L^2_T(\dot H^{s+1})}.$$

Theorem 1.2. *Let u_0 be in $\dot H^s$, with $\dfrac{1}{2} < s < \dfrac{3}{2}$. Then there exists a time T and there exists a unique solution $NS(u_0)$ such that $NS(u_0)$ belongs to $L^\infty_T(\dot H^s) \cap L^2_T(\dot H^{s+1})$.*
Moreover, let $T_*(u_0)$ be the maximal time of existence of such a solution. Then, there exists a positive constant c such that

$$T_*(u_0)\ \|u_0\|^{\sigma_s}_{\dot H^s} \geqslant c, \quad \text{with} \quad \sigma_s \stackrel{\text{def}}{=} \frac{1}{\frac{1}{2}(s-\frac{1}{2})}. \tag{6}$$

Remark 1.1. *As a by-product of the proof of Picard's Theorem, we get actually for free the following property: if the initial data is small enough (in the sense of there exists a positive constant c_0, such that $T\ \|u_0\|^{\sigma_s}_{\dot H^s} \leqslant c_0$), then a unique Navier-Stokes solution associated with it exists (locally in time, until the blow up time given by the relation (6)) and satisfies the following linear control*

$$\forall\ 0 \leqslant T \leqslant \frac{c_0}{\|u_0\|^{\sigma_s}_{\dot H^s}}, \quad \|NS(u_0)(t,\cdot)\|_{X^s_T} \leqslant 2\,\|u_0\|_{\dot H^s}. \tag{7}$$

Formula (6) invites us to consider the lower boundary, denoted by $A^{\sigma_s}_c$, of the lifespan of such a solution

$$A^{\sigma_s}_c \stackrel{\text{def}}{=} \inf\Big\{\ T_*(u_0)\|u_0\|^{\sigma_s}_{\dot H^s}\ \mid\ u_0 \in \dot H^s\ ;\ T_*(u_0) < \infty\Big\}.$$

Obviously, $A^{\sigma_s}_c$ exists and is a positive real number and we always have the formula

$$T_*(u_0)\|u_0\|^{\sigma_s}_{\dot H^s} \geqslant A^{\sigma_s}_c. \tag{8}$$

Throughout this paper, we made the assumption of blow up, which is still an open problem. More precisely, we claim the following hypothesis.

Hypothesis $\mathcal{H}$: for any $\frac{1}{2} < s < \frac{3}{2}$, a divergence-free vector field u_0 exists in $\dot H^s$ such that the lifespan $T_*(u_0)$ is finite.

Let $\mathcal{B}_\rho$ be the open ball in $\dot H^s$ defined by $\mathcal{B}_\rho = \{u_0 \in \dot H^s\ /\ \|u_0\|_{\dot H^s} < \rho\}$. Let $T_* > 0$ be a positive real number. We define a critical radius by the following formula

$$\rho_s(T_*) \stackrel{\text{def}}{=} \frac{A_c}{T_*^{\frac{1}{\sigma_s}}}.$$

Defined in this way and thanks to (8), we get an another definition of the critical radius

$$\rho_s(T_*) = \sup\{\ \rho > 0 \mid \|u_0\|_{\dot H^s} < \rho \implies T_*(u_0) > T_*\}.$$

Thanks to this definition, we define the notion of minimal blow up solution for the Navier-Stokes system.

Definition 1.3. *(minimal blow up solution)*
We say that $u = NS(u_0)$ is a minimal blow up solution if u_0 satisfies the two following assumptions:

$$\|u_0\|_{\dot H^s} = \rho_s(T_*) \quad \text{and} \quad T_*(u_0) = T_*.$$

Therefore, $u = NS(u_0)$ is a minimal blow up solution if and only if $A^{\sigma_s}_c$ is reached: $T_*(u_0)\|u_0\|^{\sigma_s}_{\dot H^s} = A^{\sigma_s}_c$.

Question: If $\rho_s(T_*)$ is finite, do some minimal blow up solutions exist ?

We will prove a stronger result: the set of initial data generating minimal blow up solutions, denoted by $\mathcal{M}_s(T_*)$, is not only a nonempty subset of $\dot{H}^s$ (which, in particular, gives the positive answer to the question) but also compact in a sense which is given in Theorem 1.3. We define the set $\mathcal{M}_s(T_*)$ as follows

$$\mathcal{M}_s(T_*) \stackrel{\text{def}}{=} \Big\{ u_0 \in \dot{H}^s \mid T_*(u_0) = T_* \quad \text{and} \quad \|u_0\|_{\dot{H}^s} = \rho_s(T_*) \Big\}.$$

Theorem 1.3. *Assuming hypothesis $\mathcal{H}$. For any finite time T_*, the set $\mathcal{M}_s(T_*)$ is non empty and compact, up to translations. This means that for any sequence $(u_{0,n})_{n\in\mathbb{N}}$ of points in the set $\mathcal{M}_s(T_*)$, a sequence $(x_n)_{n\in\mathbb{N}}$ of points of $(\mathbb{R}^3)^{\mathbb{N}}$ and a function V in $\mathcal{M}_s(T_*)$ exist such that, up to an extraction*

$$\lim_{n\to+\infty} ||u_{0,n}(\cdot + x_n) - V||_{\dot{H}^s} = 0.$$

The second result of this paper states that the blow up rate of a minimal blow up solution can be uniformely controlled since we get a priori bound of these minimal blow up solutions.

Theorem 1.4. *(Control of minimal blow up solutions)*
Assuming $\mathcal{H}$, there exists a nondecreasing function $F_s : [0, A_c^{\sigma_s}[\to \mathbb{R}^+$ with $\lim_{r\to A_c^{\sigma_s}} F_s(r) = +\infty$ such that for any divergence free vector field u_0 in $\dot{H}^s$, generating minimal blow up solution (it means $T_*(u_0)\|u_0\|_{\dot{H}^s}^{\sigma_s} = A_c^{\sigma_s}$), we have the following control on the minimal blow up solution $NS(u_0)$

$$\forall\, T < T_*(u_0),\ \|NS(u_0)\|_{X_T^s} \leqslant \|u_0\|_{\dot{H}^s}\, F_s(T^{\frac{1}{\sigma_s}}\, \|u_0\|_{\dot{H}^s}).$$

Remark 1.2. Let us point out that the quantity $T^{\frac{1}{\sigma_s}}\, \|u_0\|_{\dot{H}^s}$ is scaling invariant; which is obviously necessary.

The two previous theorems are the analogue of results, proved in the case of the Sobolev space $\dot{H}^{\frac{1}{2}}$. We shall not recall all the statements existing in the literature concerning the regularity of Navier-Stokes solutions in critical spaces, such as $\dot{H}^{\frac{1}{2}}$. We refer for instance the reader to [13] and to the article of C. Kenig et G. Koch [19], where the authors prove that NS-solutions which remain bounded in the space $\dot{H}^{\frac{1}{2}}$ do not become singular in finite time. Concerning Theorem 1.3, we were largely inspired by the article of W. Rusin and V. Šverák [29], in which the authors set up the key concept of minimal blow-up for data in Sobolev space $\dot{H}^{\frac{1}{2}}$. Firstly, they defined a critical radius $\rho_{\frac{1}{2}}$

$$\rho_{\frac{1}{2}} = \sup\big\{\rho > 0 \quad ; \quad \|u_0\|_{\dot{H}^{\frac{1}{2}}} < \rho \implies T_*(u_0) = +\infty\big\}.$$

Then, they introduced a subset $\mathcal{M}$ of $\dot{H}^{\frac{1}{2}}$, which describes the set of minimal-norm singularities (we speak about minimal norm in the sense of $\|u_0\|_{\dot{H}^{\frac{1}{2}}}$ is equal to the critical radius $\rho_{\frac{1}{2}}$)

$$\mathcal{M} = \big\{u_0 \in \dot{H}^{\frac{1}{2}} \quad ; \quad T_*(u_0) < +\infty \quad \text{and} \quad \|u_0\|_{\dot{H}^{\frac{1}{2}}} = \rho_{\frac{1}{2}}\big\}.$$

Thanks to these definitions, W. Rusin and V. Šverák proved that if there exist elements in the space $\dot{H}^{\frac{1}{2}}$ which develop singularities in finite time (we assume that blow-up occurs), then some of these elements are of minimal $\dot{H}^{\frac{1}{2}}$-norm (and thus, the set $\mathcal{M}$ is nonempty) and compact up to translations and dilations. It means that for any sequence $(u_{0,n})_{n\in\mathbb{N}}$ of points in the set $\mathcal{M}$, a sequence $(\lambda_n, x_n)_{n\in\mathbb{N}}$ and a function φ in $\mathcal{M}$ exist such that, up to an extraction, we have

$$\lim_{n\to+\infty} ||u_{0,n} - \Lambda_{\lambda_n,x_n}\varphi||_{\dot{H}^{\frac{1}{2}}} = 0.$$

Let us point out that I. Gallagher, G. Koch and F. Planchon generalize in [16] the result of W. Rusin and V. Šverák to critical Lebesgue and Besov spaces, such as L^3.

Concerning Theorem 1.4, our main source of inspiration is a result established by I. Gallagher in [14]. Given an initial data u_0 in the open ball $\mathcal{B}_{\rho_{\frac{1}{2}}}$. Then, by definition of $\rho_{\frac{1}{2}}$, $NS(u_0)$ is a global solution and thus belongs to the space $L^4(\mathbb{R}_+, \dot{H}^1)$, thanks to the important paper [15] of I. Gallagher, D. Iftimie and F. Planchon. In this way, the blow up in the $E_{\mathbb{R}_+} = L^\infty(\mathbb{R}_+, \dot{H}^{\frac{1}{2}}) \cap L^2(\mathbb{R}_+, \dot{H}^{\frac{3}{2}})$-norm does not occur. Even better: I. Gallagher proved in [14] the a priori control of the Navier-Stokes solution with data in the open ball $\mathcal{B}_{\rho_{\frac{1}{2}}}$ in the sense of there exists a nondecreasing function F defined from $[0, \rho_{\frac{1}{2}}[$ to $\mathbb{R}^+$ such that for any divergence free vector field u_0 in the open ball $\mathcal{B}_{\rho_{\frac{1}{2}}}$, we have

$$\|NS(u_0)\|_{E_{\mathbb{R}_+}} \leqslant F(\|u_0\|_{\dot{H}^{\frac{1}{2}}}).$$

Notation. We shall denote by C a constant which does not depend on the various parameters appearing in this paper, and which may change from line to line. We shall also denote sometimes $x \lesssim y$ to mean there exists an absolute constant $C > 0$ such that $x \leqslant C\, y$.

The paper is organized in the following way:
In section 2, we recall the fundamental tool of this paper : profile decomposition of a bounded sequence in $\dot{H}^s$. Then, we give the proof of the compactness of minimal blow up solutions set (Theorem 1.3) and control of of such solutions (Theorem 1.4). These two results are based on the crucial Theorem 2.2 about the lifespan of a Navier-Stokes solution associated with a bounded sequence of $\dot{H}^s$.
Section 3 is devoted to the proof of Theorem 2.2, thanks to a regularization process. Firstly, we will see that it is an immediate consequence of Lemma 3.1, which gives the structure of a Navier-Stokes solution associated with a bounded sequence of data in $\dot{H}^s$. Secondly, we will provide some helpful tools in order to prove Lemma 3.1.
In section 4, we prove Lemma 3.1, the result on which all others are based on. This section is the most technical part of the paper. It relies on classical product and paraproduct estimates, which are collected in Appendix A and B.

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2. Profiles theory, compactness result and application

This section is devoted to the proof of Theorems 1.3 and 1.4. Following I. Gallagher [14], W. Rusin and V. Šverák [29], C. Kenig and G. Koch [19] and I. Gallagher, G. Koch, F. Planchon [16], we shall use profile decomposition theory. The original motivation of this theory was the desciption of the default of compactness in Sobolev embeddings (see for instance the pionneering works of P.-L. Lions in [26], [27] and H. Brezis, J.-M. Coron in [7]. Here, we will use the theorem of P. Gérard [17], which gives, up to extractions, the structure of a bounded sequence of $\dot{H}^s$, with s between 0 and $\frac{3}{2}\cdot$ More precisely, the default of compactness in the critical Sobolev embedding $\dot{H}^s \subset L^p$ is descibed in terms of a sum of rescaled and translated orthogonal profiles, up to a small term in L^p. That was generalized to other Sobolev spaces $\dot{H}^{s,p}(\mathbb{R}^d)$ with $0 < s < \dfrac{d}{p}$ by S. Jaffard in [18], to Besov spaces by G. Koch in [23] and to general critical embeddings by H. Bahouri, A. Cohen and G. Koch in [2]. Let us notice the recent work [5] of H. Bahouri, M. Majdoud and N. Masmoudi concerning the lack of compactness of the Sobolev embedding of $H^1(\mathbb{R}^2)$ in the critical Orlicz space $\mathcal{L}(\mathbb{R}^2)$. Then profile decomposition techniques have been applied in many works of evolution problems such as the high frequency study of finite energy solutions to quintic wave equations on $\mathbb{R}^3$, by H. Bahouri and P. Gérard [4]. C. Kenig and F. Merle investigated in [20] the blow up property for the energy critical focusing non linear wave equation. Profile techniques turned out to be also a relevant tool in the study of Schrödinger equations. Notice this kind of decomposition was stated and developped, independently from [17], by F. Merle and L. Vega [28] for L^2-solutions of the critical non linear Schrödinger in $2D$, in the continuation of the work of J. Bourgain [6]. Then, S. Keraani revisited in [22] the work of H. Bahouri and P. Gérard [4] in the context of energy critical non linear Schrödinger equations. C. Kenig and F. Merle investigated in [21] the global well-posedness, scattering and blow up matter for such solutions in the focusing and radial case.

Remark 2.1. Using notation (5), we can prove easily that the L^p (as well as $\dot{H}^s$)-norm is conserved under the transformation $u \mapsto \Lambda^{\frac{3}{p}}_{\lambda,x_0} u$. It means $\|\Lambda^{\frac{3}{p}}_{\lambda,x_0} u\| = \|u\|$.

Theorem 2.1. *Let $(u_{0,n})_{n\in\mathbb{N}}$ be a bounded sequence in $\dot{H}^s$. Then, up to an extraction:*
- There exists a sequence of vectors fields, called profiles $(V^j)_{j\in\mathbb{N}}$ in $\dot{H}^s$.
- There exists a sequence of scales and cores $(\lambda_{n,j}, x_{n,j})_{n,j\in\mathbb{N}}$, such that, up to an extraction

$$\forall J \geqslant 0,\ u_{0,n}(x) = \sum_{j=0}^{J} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x) \quad \textit{with} \quad \lim_{J\to+\infty} \limsup_{n\to+\infty} \|\psi_n^J\|_{L^p} = 0, \quad \textit{and} \quad p = \frac{6}{3-2s}\cdot$$

Where $(\lambda_{n,j}, x_{n,j})_{n\in\mathbb{N},j\in\mathbb{N}^*}$ are sequences of $(\mathbb{R}_+^* \times \mathbb{R}^3)^{\mathbb{N}}$ with the following orthogonality property: for every integers (j,k) such that $j \neq k$, we have

$$\textit{either} \lim_{n\to+\infty} \Big(\frac{\lambda_{n,j}}{\lambda_{n,k}} + \frac{\lambda_{n,k}}{\lambda_{n,j}}\Big) = +\infty \quad \textit{or} \quad \lambda_{n,j} = \lambda_{n,k} \quad \textit{and} \quad \lim_{n\to+\infty} \frac{|x_{n,j} - x_{n,k}|}{\lambda_{n,j}} = +\infty.$$

Moreover, for any J in $\mathbb{N}$, we have the following orthogonality property

$$\|u_{0,n}\|^2_{\dot{H}^s} = \sum_{j=0}^{J} \|V^j\|^2_{\dot{H}^s} + \|\psi_n^J\|^2_{\dot{H}^s} + \circ(1), \quad \textit{when} \quad n \to +\infty. \tag{9}$$

A first application of this, is Theorem 2.2 about the lifespan of a NS-solution associated with bounded data in $\dot{H}^s$. The proof of it will be given in section 3.

Theorem 2.2. *Let $(u_{0,n})$ be a bounded sequence of initial data in $\dot{H}^s$ such that its profiles decomposition is given by*

$$u_{0,n}(x) = \sum_{j=0}^{J} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x) \quad \textit{with} \quad \lim_{J\to+\infty} \limsup_{n\to+\infty} \|\psi_n^J\|_{L^p} = 0.$$

Let us define $\mathcal{J}_1$ as the subset of indices j in $\mathbb{N}$, such that the profile V^j is non-zero and such that the associated scale $\lambda_{n,j}$ is identically equal to 1.

$$\textit{If } \mathcal{J}_1 = \emptyset, \ \textit{then} \ \liminf_{n\to+\infty} T_*(u_{0,n}) = +\infty.$$
$$\textit{If } \mathcal{J}_1 \neq \emptyset, \ \textit{then} \ \liminf_{n\to+\infty} T_*(u_{0,n}) \geqslant \inf_{j\in\mathcal{J}_1} T_*(V^j).$$

Remark 2.2. Let us point out some facts. Firstly, if $T_*(V^j) = +\infty$ for any j, then $\liminf_{n\to+\infty} T_*(u_{0,n}) = +\infty$. Secondly, in the case where $\mathcal{J}_1$ is non empty, the quantity $\inf_{j\in\mathcal{J}_1} T_*(V^j)$ exists and obviously, if $|\mathcal{J}_1|$ is finite, we get immediately that $\inf_{j\in\mathcal{J}_1} T_*(V^j) = \min_{j\in\mathcal{J}_1} T_*(V^j)$. In the case where $|\mathcal{J}_1|$ is infinite, we get the same conclusion. Indeed, by vertue of (9), the serie $\sum_{j\geqslant 0} \|V^j\|^2_{\dot{H}^s}$ is summable (*a fortiori* if we consider in the summation integers belonging to $\mathcal{J}_1$), and thus $\lim_{j\to+\infty} \|V^j\|_{\dot{H}^s} = 0$. Thanks to Inequality (6), we deduce that $\lim_{j\to+\infty} T_*(V^j) = +\infty$ and thus

$$\inf_{j\in\mathcal{J}_1} T_*(V^j) > 0 \quad \text{and} \quad \inf_{j\in\mathcal{J}_1} T_*(V^j) = \min_{j\in\mathcal{J}_1} T_*(V^j).$$

This result gives us an important information: whenever a sequence of initial data which satisfies profiles hypothesis (it means a bounded sequence in $\dot{H}^s$), we get an information on the lifespan of the NS-solution associated with such a sequence of initial data: it mainly depends on the lifespan of profiles with a constant scale. Note that the orthogonality property on scales and cores in Theorem 2.1 implies either the scales are different (in the sense that $\lim_{n\to+\infty}\Big(\frac{\lambda_{n,j}}{\lambda_{n,k}}+\frac{\lambda_{n,k}}{\lambda_{n,j}}\Big)=+\infty$) or the scales are the same ($\lambda_{n,j}=\lambda_{n,k}$), equal to a constant, and the cores go away from one another, in the sense that $\lim_{n\to+\infty}\frac{|x_{n,j}-x_{n,k}|}{\lambda_{n,j}}=+\infty$. In the last case where scales are equal to a constant, we shall assume that it is one, up to rescaling profiles by a fixed constant.
Theorem 2.2 has a key role in the proof of the compactness Theorem 1.3: the set $\mathcal{M}_s(T_*)$, recalled below, is non empty and compact, up to translations.

$$\mathcal{M}_s(T_*):=\Big\{u_0\in\dot{H}^s \mid T_*(u_0)=T_* \quad\text{and}\quad \|u_0\|_{\dot{H}^s}=\rho_s(T_*)\Big\}.$$

2.1. **Proof of the compactness Theorem 1.3.**

Proof. By definition of $A_c^{\sigma_s}$, we consider a minimizing sequence $(u_{0,n})_{n\geqslant 0}$ such that

$$\lim_{n\to+\infty} T_*(u_{0,n})\,\|u_{0,n}\|_{\dot{H}^s}^{\sigma_s}=A_c^{\sigma_s}.$$

Up to a rescaling process, we can assume that the minimizing sequence $(u_{0,n})_{n\geqslant 0}$ satisfies

$$\lim_{n\to+\infty}\|u_{0,n}\|_{\dot{H}^s}=\rho_s(T_*)\quad\text{and}\quad T_*(u_{0,n})=T_*. \tag{10}$$

Indeed, consider the sequence $(v_{0,n})_{n\geqslant 0}$ defined as

$$v_{0,n}(x)\stackrel{\text{def}}{=}\Big(\frac{T_*(u_{0,n})}{T_*}\Big)^{\frac12}u_{0,n}\Big(\Big(\frac{T_*(u_{0,n})}{T_*}\Big)^{\frac12}x\Big).$$

The reader notices that the Navier-Stokes solution associated with such a sequence $(v_{0,n})$ has a lifespan equal to T_*. As $\|v_{0,n}\|_{\dot{H}^s}^{\sigma_s}=\Big(\frac{T_*(u_{0,n})}{T_*}\Big)\|u_{0,n}\|_{\dot{H}^s}^{\sigma_s}$, it seems clear now we can assume (10), by vertue of definition of $\rho_s(T_*)$. As defined, $(u_{0,n})_{n\geqslant 0}$ is a sequence of points of the set $\mathcal{M}_s(T_*)$; it is a bounded sequence in $\dot{H}^s$ and thus we can apply Theorem 2.1. Taking limit when $n\to+\infty$ in (9), we get

$$\forall J\geqslant 0,\quad \rho_s^2(T_*)\geqslant\sum_{j=0}^{J}\|V^j\|_{\dot{H}^s}^2.$$

Let us assume that there are two non-zero profiles at least. Then we should have

$$\forall j\in\{0,\cdots,J\},\ \|V^j\|_{\dot{H}^s}^2<\rho_s^2(T_*).$$

By definiton of $\rho_s(T_*)$, it means all profiles V^j generate solutions whose lifespan satisfies

$$T_*(V^j)>T_*,\ \forall j\in\{0,\cdots,J\} \tag{11}$$

As $T_*(u_{0,n})=T_*<\infty$ for any n, Theorem 2.2 implies that $\mathcal{J}_1\neq\emptyset$: there exists at least one profile with constant scale. Moreover, thanks to Remark 2.2, we have $\inf_{j\in\mathcal{J}_1}T_*(V^j)=\min_{j\in\mathcal{J}_1}T_*(V^j)$. Combining this with Relation (11) implies that

$$\widetilde{T}\stackrel{\text{def}}{=}\inf_{j\in\mathcal{J}_1}T_*(V^j)>T_*.$$

By hypothesis on $(u_{0,n})_{n\in\mathbb{N}}$ and thanks to Theorem 2.2, we get a contradiction, since we have

$$\liminf_{n\to+\infty}T_*(u_{0,n})=T_*\geqslant\widetilde{T}>T_*.$$

It means there exists an integer j_0 such that the profile, V^{j_0} has a lifespan which satisfies $T_*^{j_0} \leqslant T_*$. In particular, by definition of $\rho_s(T_*)$, it implies that $\|V^{j_0}\|^2_{\dot{H}^s} \geqslant \rho_s^2(T_*)$. And, thanks to the orthogonal property of the $\dot{H}^s$-norm (9), we deduce the equality

$$\|V^{j_0}\|^2_{\dot{H}^s} = \rho_s^2(T_*).$$

Now, we have just to check that $T_* = T_*^{j_0}$. We have already proved a first inequality: $T_*^{j_0} \leqslant T_*$. The other way is given by (8): we have always the following relation: $T_*^{j_0}\|V^{j_0}\|^{\sigma_s}_{\dot{H}^s} \geqslant A_c^{\sigma_s}$. Thanks to the result $\|V^{j_0}\|^{\sigma_s}_{\dot{H}^s} = \rho_s^{\sigma_s}(T_*) = \dfrac{A_c^{\sigma_s}}{T_*}$, we get the second inequality: $T_*^{j_0} \geqslant T_*$. Thus, the set $\mathcal{M}_s(T_*)$ is non empty and thus, there exists some minimal Navier-Stokes solutions. The compactness of the set $\mathcal{M}_s(T_*)$ is a consequence of the above work. Thanks to (9) and $\|V^{j_0}\|_{\dot{H}^s} = \rho_s(T_*)$, we infer that

$$\forall j \neq j_0, \;\; V^j = 0 \quad \text{and} \quad \lim_{n\to+\infty} \|\psi_n^J\|^2_{\dot{H}^s} = 0.$$

The above assumption implies in particular that $j_0 \in \mathcal{J}_1$. Indeed, if $j_0 \notin \mathcal{J}_1$, then $\mathcal{J}_1 = \emptyset$ and thus we should have $T_* = +\infty$, which is absurd. As a result, there exists a unique integer $j_0 \in \mathcal{J}_1$, such that

$$u_{0,n}(x) = V^{j_0}(x - x_{n,j_0}) + \psi_n^J(x).$$

The property $\lim\limits_{n\to+\infty} \|\psi_n^l\|^2_{\dot{H}^s} = 0$ implies $\lim\limits_{n\to+\infty} \|u_{0,n}(\cdot + x_{j_0,n}) - V^{j_0}\|_{\dot{H}^s} = 0.$ □

2.2. **Proof of Theorem 1.4.**

Proof. Let us consider a critical element $u = NS(u_0) : T_*^{\frac{1}{\sigma_s}}(u_0)\|u_0\|_{\dot{H}^s} = A_c$. By vertue of a rescaling, we can asume that $\|u_0\|_{\dot{H}^s} = 1$ and thus $T_*^{\frac{1}{\sigma_s}}(u_0) = A_c$. Let us introduce the following set

$$\mathcal{N}_T^s \overset{\text{def}}{=} \Big\{ \|NS(u_0)\|_{X_T} \;\Big|\; u_0 \text{ in } \dot{H}^s \text{ such that } \|u_0\|_{\dot{H}^s} = 1 \quad \text{and} \quad T < A_c^{\sigma_s} \Big\}.$$

Theorem 1.3 claims that the set $\mathcal{N}_T^s$ is nonempty. The aim is to prove that $\sup \mathcal{N}_T^s$ is finite for any T. If not, a sequence $(u_{0,n})_{n\geqslant 0}$ in $\dot{H}^s$ exists, such that for any $T < T_*(u_{0,n})$, we have

$$\|u_{0,n}\|_{\dot{H}^s} = 1, \quad T_*(u_{0,n}) = A_c^{\sigma_s} \quad \text{and} \quad \lim_{n\to+\infty} \|NS(u_{0,n})\|_{X_T} = \infty. \tag{12}$$

By hypothesis, the sequence $(u_{0,n})_{n\geqslant 0}$ belongs to the set $\mathcal{M}_s(T_*)$. Therefore, there exist a sequence of cores $(x_n)_{n\in\mathbb{N}}$ and a function V in $\mathcal{M}_s(T_*)$ such that, up to an extraction:

$$\lim_{n\to+\infty} ||u_{0,n}(\cdot + x_n) - V||_{\dot{H}^s} = 0. \tag{13}$$

We can prove easily that, for any $T < T_*(V)$:

$$NS(u_{0,n}(\cdot + x_n)) = NS(V) + R_n \text{ with } \lim_{n\to+\infty} \|R_n\|_{X_T} = 0 \tag{14}$$

Indeed, we define

$$R_{0,n} \overset{\text{def}}{=} u_{0,n}(\cdot + x_n) - V.$$

Because of (13), the sequence $(R_{0,n})_{n\geqslant 0}$ converges to 0 in $\dot{H}^s$-norm, for n large enough. Moreover, the error term R_n satifies the following perturbed Navier-Stokes system

$$\left\{ \begin{array}{rcl} \partial_t R_n + R_n \cdot \nabla R_n - \Delta R_n + R_n \cdot \nabla NS(V) + NS(V) \cdot \nabla R_n & = & -\nabla p \\ \operatorname{div} R_n & = & 0 \\ R_{n|t=0} & = & R_{0,n}. \end{array} \right. \tag{15}$$

Applying forthcoming Theorem 6.1, we infer that, for any $T < T_*(V)$ and for n large enough

$$\|NS(u_{0,n}(\cdot + x_n))\|_{X_T} \leqslant \|NS(V)\|_{X_T} + \circ(1).$$

As $\|NS(u_{0,n}(\cdot + x_n))\|_{X_T} = \|NS(u_{0,n})\|_{X_T}$, we take the limit when $n \to +\infty$ in the above inequality and thus we get a contradiction with the assumption. □

3. Proof of Theorem 2.2 and tool box for Lemma 3.1

All the previous results are based on Theorem 2.2. In this section, we prove this theorem, which relies on Lemma 3.1. This last one gives the structure of the Navier-Stokes solution associated with an initial data which has a profile decomposition. In others words, we wonder if, given the profile decomposition of a sequence of data, we get a similar decomposition on the Navier-Stokes solution itself. Lemma 3.1 gives a positive answer.
Let us recall to the reader that this question has already been studied by I. Gallagher in [14] in the case of initial data in the Sobolev space $\dot{H}^{\frac{1}{2}}$ and the same author with G. Koch, F. Planchon [16] in others critical spaces (e.g scaled invariant under the Navier-Stokes transformation). In our case, the difficulty is that the homogeneous Sobolev space $\dot{H}^s$ is not a scale invariant space under the natural scaling of the Navier-Stokes equation. To overcome this issue, the method consists in cutting off frequencies of profiles [4] (such profiles will have the useful property to belong to any $\dot{H}^s$, for any s). In particular, profiles scaled by 0 (resp. ∞) will tend to 0 in some Sobolev spaces (more precisely in $\dot{H}^{s_1}$ with $s_1 < s$), (resp. $\dot{H}^{s_2}$ with $s_2 > s$) and therefore, will not perturb the profile decomposition of the NS-solution.

3.1. **Key Lemma and application.** Let $(u_{0,n})_{n\geqslant 0}$ be a bounded sequence of initial data in $\dot{H}^s$. Thanks to Theorem 2.1, $(u_{0,n})_{n\geqslant 0}$ can be written as follows, up to an extraction

$$u_{0,n}(x) = \sum_{j=0}^{J} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x).$$

By vertue of orthogonality of scales and cores given by Theorem 2.1, we sort profiles according to their scales

$$u_{0,n}(x) = \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} V^j(x - x_{n,j}) + \sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x) \tag{16}$$

where for any $j \in \mathcal{J}_1$, for any $n \in \mathbb{N}$, $\lambda_{n,j} \equiv 1$.

We claim we have the following structure lemma of the Navier-Stokes solutions, which proof will be provided in section 4. This lemma highlights the specific role of profiles with constant-scales.

Lemma 3.1. *(Profile decomposition of the Navier-Stokes solution)*
Let $(u_{0,n})_{n\geqslant 0}$ be a bounded sequence of initial data in $\dot{H}^s$ which profile decomposition is given by

$$u_{0,n}(x) = \sum_{j=0}^{J} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x).$$

Then, u is a Navier-Stokes solution associated with the bounded sequence $u_{0,n}$ (e.g $u = NS(u_{0,n})$) if and only if the error term R_n^J defined by $R_n^J \stackrel{\text{def}}{=} NS(u_{0,n}) - U_n^{\text{app,J}}$ is a solution of the below perturbed Navier-Stokes equation

$$\left\{\begin{array}{ccc} \partial_t R_n^J + R_n^J.\nabla R_n^J - \Delta R_n^J + R_n^J \cdot \nabla U_n^{\text{app,J}} + U_n^{\text{app,J}}.\nabla R_n^J & = & -F_n^J - \nabla p_n^J \\ \text{div } R_n^J & = & 0 \\ R_{n|t=0}^J & = & 0. \end{array}\right. \tag{17}$$

where F_n^J is a forcing term which will be explicitely detailed in (26) and

$$U_n^{\text{app,J}}(t,x) \stackrel{\text{def}}{=} \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t, x - x_{n,j}) \;+\; e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big).$$

Moreover, the lifespan τ_n^J of the error term R_n^J satisifies

$$\forall \varepsilon > 0,\ \exists J \geqslant 0\ \exists n_J \geqslant 0\ \forall n \geqslant n_J, \quad \tau_n^J \geqslant \inf_{j \in J_1} T_*(V^j) - \varepsilon.$$

Proof of Theorem 2.2. Clearly, Theorem 2.2 is an immediate consequence of Lemma 3.1. Assume Lemma 3.1 is proved. On the one hand, if there is no non zero profile with constant scale (e.g $\mathcal{J}_1 = \emptyset$), the "profile decomposition" of the solution in Lemma 3.1 implies that $\liminf\limits_{n\to+\infty} T_*(u_{0,n}) = +\infty$. On the other hand, if $\mathcal{J}_1 \neq \emptyset$, the lifespan of sequence $NS(u_{0,n})$ is given by the lifespan of profiles, scaled by the constant 1 and $T_*(u_{0,n}) \geqslant \inf\limits_{j\in\mathcal{J}_1} T_*(V^j)$. This ends up the proof of Theorem 2.2.

3.2. **Tool box.** In this subsection, we recall some basic facts about homogeneous Besov spaces and we prove some properties we need to the proof of Lemma 3.1. We refer the reader to [1], from page 63, for a detailed presentation of the theory and analysis of homogeneous Besov spaces.

Definition 3.1. *Let s be in $\mathbb{R}$, (p,r) in $[1,+\infty]^2$ and u in $\mathcal{S}'$. A tempered distribution u is an element of the Besov space $\dot{B}^s_{p,r}$ if u satifies*

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \Big(\sum_{j\in\mathbb{Z}} 2^{jrs}\ ||\dot{\Delta}_j u||^r_{L^p}\Big)^{\frac{1}{r}} < \infty,$$

where $\dot{\Delta}_j$ is a frequencies localization operator (called Littlewood-Paley operator), defined by

$$\dot{\Delta}_j u(\xi) \stackrel{\text{def}}{=} \mathcal{F}^{-1}\big(\varphi(2^{-j}|\xi|)\widehat{u}(\xi)\big),$$

with $\varphi \in \mathcal{D}([\frac{1}{2},2])$, such that $\sum\limits_{j\in\mathbb{Z}} \varphi(2^{-j}t) = 1$, for any $t > 0$.

Remark 3.1. We have the embedding $\dot{H}^s \subset \dot{B}^s_{2,2}$. These spaces coincide if $s < \frac{3}{2}$.

The first thing we have to notice is the following: given a bounded sequence of data in $\dot{H}^s$ (thus we get a profile decomposition of this sequence), Theorem 2.1 implies that the term $\psi_n^J(x)$, (which is bounded in $\dot{H}^s$), satisfies:

$$\lim_{J\to+\infty} \limsup_{n\to+\infty} \|\psi_n^J\|_{L^p} = 0.$$

In fact, thanks to an interpolation argument, we can prove that the remaining term ψ_n^J tends to 0 in certain Besov spaces. That is the point in the following proposition.

Proposition 3.2. *For any $0 < \theta < 1$, let p_θ be a positive real number given by the interpolation relation*

$$\frac{1}{p_\theta} = \frac{\theta}{p} + \frac{1-\theta}{2}\cdot$$

Then, under the same hypothesis of Theorem 2.1, we have:

$$\lim_{J\to+\infty} \limsup_{n\to+\infty} \|\psi_n^J\|_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}} = 0\cdot$$

Proof. Interpolation inequality in the Lebesgue spaces and multiplication by the factor $2^{js(1-\theta)}$ give

$$2^{js(1-\theta)}\|\dot{\Delta}_j\psi_n^J\|_{L^{p_\theta}} \leqslant \|\dot{\Delta}_j\psi_n^J\|^\theta_{L^p}\ \big(2^{js}\|\dot{\Delta}_j\psi_n^J\|_{L^2}\big)^{1-\theta}.$$

Applying Hölder's inequality in the above expression, we get

$$\|\psi_n^J\|_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}} \leqslant \|\psi_n^J\|^\theta_{\dot{B}^0_{p,p}}\ \ \|\psi_n^J\|^{1-\theta}_{\dot{B}^s_{2,2}}.$$

Because p is greater than 2, L^p is continuously included in $\dot{B}^0_{p,p}$. Remark 3.1 leads to

$$\|\psi_n^J\|_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}} \leqslant \|\psi_n^J\|^\theta_{L^p}\|\psi_n^J\|^{1-\theta}_{\dot{H}^s}. \tag{18}$$

By vertue of Theorem 2.1, we get the result. □

Let us come back to the profile decomposition of the sequence $(u_{0,n})_{n\geqslant 0}$ and introduce some notations. Let $\eta > 0$ be the parameter of rough cutting off frequencies. We define by $u_\eta(x)$ and $u_{^c\eta}(x)$ the elements which Fourier transform is given by

$$\widehat{u_\eta}(\xi) = \widehat{u}(\xi) 1_{\{\frac{1}{\eta}\leqslant|\xi|\leqslant\eta\}} \quad \text{and} \quad \widehat{u_{^c\eta}}(\xi) = \widehat{u}(\xi)\big(1 - 1_{\{\frac{1}{\eta}\leqslant|\xi|\leqslant\eta\}}\big). \tag{19}$$

From the profiles decomposition (16), we infer, thanks to the orthogonality property of scales, that among profiles V^j such that j belongs to ${}^c\mathcal{J}_1$, there are profiles with small scales ($j \in \mathcal{J}_0$) and large scales ($j \in \mathcal{J}_\infty$). These profiles are cut (according to the parameter η), with respect to notations (19) and we get

$$\begin{aligned} u_{0,n}(x) &= \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} V^j(x - x_{n,j}) + \sum_{\substack{j\in\mathcal{J}_0\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_\eta(x) + \sum_{\substack{j\in\mathcal{J}_\infty\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_\eta(x) + \psi^J_{n,\eta}(x) \\ &\text{where} \quad \psi^J_{n,\eta}(x) \stackrel{\text{def}}{=} \sum_{\substack{j\in\mathcal{J}^c_1\equiv\mathcal{J}_0\cup\mathcal{J}_\infty\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_{^c\eta}(x) \,+\, \psi^J_n(x), \end{aligned} \tag{20}$$

with for any $j \in \mathcal{J}_0$, $\lim_{n\to+\infty} \lambda_{n,j} = 0$ and for any $j \in \mathcal{J}_\infty$, $\lim_{n\to+\infty} \lambda_{n,j} = +\infty$.

Firstly, we check the remaining term $\psi^J_{n,\eta}$ is still small in $\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}$-norm, in the following sense. That is the point of the proposition below.

Proposition 3.3. *Let $0 < \theta < 1$. Under the interpolation relation $\dfrac{1}{p_\theta} = \dfrac{\theta}{p} + \dfrac{1-\theta}{2}$, we have*

$$\lim_{J\to+\infty} \lim_{\eta\to+\infty} \limsup_{n\to+\infty} \|\psi^J_{n,\eta}\|_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}} = 0.$$

Proof. Let $0 < \theta < 1$. By definition of $\psi^J_{n,\eta}$ and thanks to $(a + b)^2 \lesssim a^2 + b^2$, we have

$$\|\psi^J_{n,\eta}\|^2_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}} \lesssim \Big\| \sum_{\substack{j\in\mathcal{J}^c_1\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_{^c\eta}(x) \Big\|^2_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}} + \|\psi^J_n\|^2_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}}. \tag{21}$$

The embedding $\dot{H}^s \subset \dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}$ and the orthogonality of scales and cores imply

$$\begin{aligned} \|\psi^J_{n,\eta}\|^2_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}} &\lesssim \Big\| \sum_{\substack{j\in\mathcal{J}^c_1\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_{^c\eta}(x) \Big\|^2_{\dot{H}^s} + \|\psi^J_n\|^2_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}} \\ &\lesssim \Big(\sum_{\substack{j\in\mathcal{J}^c_1\\ j\leqslant J}} \big\|\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_{^c\eta}(x)\big\|^2_{\dot{H}^s}\big\|^2_{\dot{H}^s} + \circ(1) \Big) + \|\psi^J_n\|^2_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}}. \end{aligned} \tag{22}$$

By scaling invariance of the norm $\dot{H}^s$ under the transformation $u \mapsto \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} u$, we get

$$\|\psi^J_{n,\eta}\|^2_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}} \lesssim \Big(\sum_{j=0}^{\infty} \big\|V^j_{^c\eta}(x)\big\|^2_{\dot{H}^s} + \circ(1) \Big) + \|\psi^J_n\|^2_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}}, \text{ when } \quad n \to +\infty.$$

For any $j \geqslant 0$, the term $\big\|V^j_{^c\eta}(x)\big\|^2_{\dot{H}^s}$ tends to 0 for η large enough, by Lebesgue Theorem. Therefore, applying Lebesgue Theorem once again, we infer that $\lim_{\eta\to+\infty} \sum_{j=0}^{\infty} \big\|V^j_{^c\eta}(x)\big\|^2_{\dot{H}^s} = 0$. As a result, we take in first the upper limit of $\|\psi^J_{n,\eta}\|^2_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}}$, when $n \to +\infty$. Then, we take the limit for $\eta \to +\infty$ and at the last, for $J \to +\infty$. Thanks to Proposition 3.2, Proposition 3.3 is proved. □

As it was already mentionned previously, the point of such rough cutting off in frequencies is that profiles which are supported in the annulus $1_{\{\frac{1}{\eta} \leqslant |\xi| \leqslant \eta\}}$, belong to the Sobolev spaces $\dot{H}^s$, for any $s > 0$. In particular, we can look at such profiles in the Sobolev spaces such as $\dot{H}^{s_1}$ with $s_1 < s$ and $\dot{H}^{s_2}$ with $s_2 > s$. That is the point in the following proposition: according to the size of the scale (either small j in $\mathcal{J}_0$ or large j in $\mathcal{J}_\infty$), profiles, trapped in the annulus, behave theirselves as "remaining terms", seen from the point of view of solving Navier-Stokes.

Proposition 3.4.

$$\textit{For any } \eta > 0,\ s_1 < s,\ \textit{and } j \in \mathcal{J}_0,\ \textit{e.g} \lim_{n\to+\infty} \lambda_{n,j} = 0,\ \textit{then} \lim_{n\to+\infty} \Big\| \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_\eta(x) \Big\|_{\dot{H}^{s_1}} = 0.$$

$$\textit{For any } \eta > 0,\ s_2 > s,\ \textit{and } j \in \mathcal{J}_\infty,\ \textit{e.g} \lim_{n\to+\infty} \lambda_{n,j} = +\infty,\ \textit{then} \lim_{n\to+\infty} \Big\| \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_\eta(x) \Big\|_{\dot{H}^{s_2}} = 0.$$

Proof. Let $s_1 < s$. Let $j \in \mathcal{J}_0$ and $\eta > 0$. Definition of $\dot{H}^{s_1}$-norm and a variable change yield

$$\begin{aligned} \Big\| \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_\eta(x) \Big\|^2_{\dot{H}^{s_1}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\xi|^{2s_1} \Big| {\lambda_{n,j}}^{3(1-\frac{1}{p})} \widehat{V^j_\eta}(\lambda_{n,j}\xi) \Big|^2 d\xi \\ &= \lambda_{n,j}^{2(s-s_1)} \int_{\mathbb{R}^3} |\xi|^{2s_1} \, |\widehat{V^j_\eta}(\xi)|^2 d\xi. \end{aligned} \tag{23}$$

Let us introduce the factor $|\xi|$. The hypothesis of the ring implies that

$$\begin{aligned} \Big\| \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_\eta(x) \Big\|^2_{\dot{H}^{s_1}} &= {\lambda_{n,j}}^{2(s-s_1)} \int_{\mathbb{R}^3} |\xi|^{2s} \, |\widehat{V^j_\eta}(\xi)|^2 \frac{1}{|\xi|^{2(s-s_1)}} d\xi \\ &\leqslant (\eta \ \lambda_{n,j})^{2(s-s_1)} \|V^j\|^2_{\dot{H}^s}. \end{aligned} \tag{24}$$

As $\lambda_{n,j}$ tends to 0; this proves the first part of the proposition. The second part relies on similar arguments and thus the proof is omitted. □

4. Proof of Lemma 3.1

Given a bounded sequence $(u_{0,n})$ in $\dot{H}^s$ which profile decomposition is given by Theorem 2.1, we search sequences associated solutions $NS(u_{0,n})$, under the form of

$$NS(u_{0,n}) = U_n^{\text{app,J}} + R^J_n, \quad \text{where}$$

$$U_n^{\text{app,J}} \overset{\text{def}}{=} \sum_{\substack{j \in \mathcal{J}_1 \\ j \leqslant J}} NS(V^j)(t, \cdot - x_{n,j}) + e^{t\Delta} \Big(\sum_{\substack{j \in \mathcal{J}_1^c \\ j \leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi^J_n(x) \Big), \tag{25}$$

Note that if $\mathcal{J}_1 = \emptyset$, the the approximation term $U_n^{\text{app,J}}$ is reduced to the linear part

$$U_n^{\text{app,J}} = e^{t\Delta} \Big(\sum_{\substack{j \in \mathcal{J}_1^c \\ j \leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi^J_n(x) \Big).$$

Plugging this decomposition into the Navier-Stokes equation leads to the following perturbed equation on the error term R^J_n

$$\left\{ \begin{array}{rcl} \partial_t R^J_n + R^J_n.\nabla R^J_n - \Delta R^J_n + R^J_n \cdot \nabla U_n^{\text{app,J}} + U_n^{\text{app,J}}.\nabla R^J_n &=& -F^J_n - \nabla p^J_n \\ \text{div } R^J_n &=& 0 \\ R^J_{n|t=0} &=& 0. \end{array} \right. \tag{26}$$

where the forcing term F_n^J is given by $F_n^J = \sum_{\ell=1}^{4} F_n^{J,\ell}$, with

$$
\begin{aligned}
& F_n^{J,1} = \sum_{0\leqslant j,k\leqslant \mathcal{J}_1; j\neq k} NS(V^j)(t,\cdot - x_{n,j})\cdot \nabla NS(V^k)(t,\cdot - x_{n,k}),\\
& F_n^{J,2} = e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big)\cdot \nabla\Big(e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big)\Big),\\
& F_n^{J,3} = e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big)\cdot \nabla\Big(\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot - x_{n,j})\Big),\\
& F_n^{J,4} = \Big(\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot - x_{n,j})\Big)\cdot \nabla\Big(e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big)\Big).
\end{aligned}
\tag{27}
$$

Let us admit for a while the two following propositions.

Proposition 4.1. *With notations (37), the sequence $U_n^{\text{app,J}}$ is bounded in the space X_T^s, uniformly in J and η,*

$$
\|U_n^{\text{app,J}}\|_{X_T^s} < \infty, \quad \forall T < \widetilde{T} \stackrel{\text{def}}{=} \inf_{j\in\mathcal{J}_1} T_*(V^j).
$$

Once again, we use the convention that $\inf_{j\in\mathcal{J}_1} T_*(V^j) = +\infty$ if $\mathcal{J}_1$ is empty. Let us admit for a while the following proposition.

Proposition 4.2.

$$
\lim_{J\to+\infty} \limsup_{n\to+\infty} \|F_n^J\|^2_{L^2_T(\dot H^{s-1})} = 0.
$$

Completion of the proof of Lemma 3.1. Let $\varepsilon_0 > 0$. Let T_0 be the time defined by

$$
T_0 \stackrel{\text{def}}{=} \sup\big\{0 < T < \widetilde{T} \mid \|R_n^J(t)\|^2_{L^\infty_T(\dot H^s)} \leqslant \varepsilon_0\big\}.
$$

Therefore, for any $T < T_0 \leqslant \widetilde{T}$, Theorem 6.1 implies

$$
\|R_n^J\|^2_{X_T^s} \lesssim \|F_n^J\|^2_{L^2_T(\dot H^{s-1})} \exp\Big(\varepsilon_0^{\frac{2}{2s-1}}\, \widetilde{T} + \widetilde{T}^{s-\frac12}\, \|U_n^{\text{app,J}}\|^2_{X_T^s} + \widetilde{T}\, \|U_n^{\text{app,J}}\|^{\frac{4}{2s-1}}_{L^\infty_T(\dot H^s)}\Big).
\tag{28}
$$

Combining Propositions 4.1 and 4.2, Lemma 3.1 is proved. Therefore, to complete the proof, we shall prove the two above propositions.

Proof of Proposition 4.1. By definition of $U_n^{\text{app,J}}$ and vertue of $(a+b)^2 \leqslant 2\,(a^2+b^2)$, we have

$$
\|U_n^{\text{app,J}}\|^2_{X_T^s} \leqslant 2\Big(\Big\|\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot - x_{n,j})\Big\|^2_{X_T^s} + \Big\|e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big)\Big\|^2_{X_T^s}\Big).
\tag{29}
$$

Let us focus for a moment on the heat term $e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big)$. It is well-known that an $\dot H^s$-energy estimate on the heat equation implies that $\big\|e^{t\Delta}u\big\|^2_{X_T^s} \leqslant \|u_0\|^2_{\dot H^s}$, for any u solution associated with data u_0 in $\dot H^s$. As a result, we get

$$
\Big\|e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big)\Big\|^2_{X_T^s} \leqslant \Big\|\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j(x) + \psi_n^J(x)\Big\|^2_{\dot H^s}.
$$

Therefore, profile decomposition yields, up to triangular and Young's inequalites

$$\|e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}}\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}V^j(x)+\psi_n^J(x)\Big)\|^2_{X^s_T}\leqslant\Big\|u_{0,n}-\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}V^j(\cdot-x_{n,j})\Big\|^2_{\dot H^s}$$

$$\leqslant 2\left\|u_{0,n}\right\|^2_{\dot H^s}+2\Big\|\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}V^j(\cdot-x_{n,j})\Big\|^2_{\dot H^s}.$$

Let us admit for a while the following statement

$$\forall\eta>0,\quad\Big\|\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}V^j(\cdot-x_{n,j})\Big\|^2_{\dot H^s}=\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}\left\|V^j\right\|^2_{\dot H^s}+\circ(1),\quad\text{when}\quad n\to+\infty.\tag{30}$$

Thanks to the orthogonality relation (9), the term $\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}\left\|V^j\right\|^2_{\dot H^s}$ satisfies $\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}\left\|V^j\right\|^2_{\dot H^s}\leqslant\left\|u_{0,n}\right\|^2_{\dot H^s}+\circ(1)$, for n large enough. As a result,

$$\forall\eta>0,\quad\Big\|e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}}\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}V^j(x)+\psi_n^J(x)\Big)\Big\|^2_{X^s_T}\lesssim\left\|u_{0,n}\right\|^2_{\dot H^s}+\circ(1),\quad\text{when}\quad n\to+\infty.\tag{31}$$

Now, let us come back to (29). Thanks to the previous estimate (31), we infer that

$$\forall\eta>0,\quad\|U_n^{\text{app,J}}\|^2_{X^s_T}\lesssim\Big\|\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}NS(V^j)(t,\cdot-x_{n,j})\Big\|^2_{X^s_T}+\left\|u_{0,n}\right\|^2_{\dot H^s}+\circ(1),\quad\text{when}\quad n\to+\infty.$$

We admit for a while the following statement, for any $T<\widetilde{T}\stackrel{\text{def}}{=}\inf_{j\in\mathcal{J}_1}T_*(V^j)$ and $\eta>0$.

$$\Big\|\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}NS(V^j)(t,\cdot-x_{n,j})\Big\|^2_{X^s_T}\leqslant\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}\left\|NS(V^j)(t,\cdot)\right\|^2_{X^s_T}+\circ(1),\quad\text{when}\quad n\to+\infty.\tag{32}$$

Therefore, we have for any $T<\widetilde{T}\stackrel{\text{def}}{=}\inf_{j\in\mathcal{J}_1}T_*(V^j)$

$$\forall\eta>0,\quad\|U_n^{\text{app,J}}\|^2_{X^s_T}\leqslant C\Big(\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}\left\|NS(V^j)\right\|^2_{X^s_T}+\left\|u_{0,n}\right\|^2_{\dot H^s}+\circ(1)\Big),\quad\text{when}\quad n\to+\infty.$$

As $NS(V^j_\eta)$ solves NS-equation with initial data V^j_η belonging to $\dot H^s$ and since the time T is far away from the blow up time, we infer that each term in the right-hand side is bounded, uniformly in J and η. Now let us prove (30). Clearly we have, thanks to the translations invariance of the $\dot H^s$-norm

$$\Big\|\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}V^j(\cdot-x_{n,j})\Big\|^2_{\dot H^s}=\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}\left\|V^j(\cdot-x_{n,j})\right\|^2_{\dot H^s}+2\sum_{\substack{(j,k)\in\mathcal{J}_1\times\mathcal{J}_1\\ j\neq k}}\Big(V^j(\cdot-x_{n,j})\,\big|\,V^k(\cdot-x_{n,k})\Big)_{\dot H^s}$$

$$=\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}}\left\|V^j\right\|^2_{\dot H^s}+2\sum_{\substack{(j,k)\in\mathcal{J}_1\times\mathcal{J}_1\\ j\neq k}}\Big(|D|^sV^j(\cdot-x_{n,j})\,\big|\,|D|^sV^k(\cdot-x_{n,k})\Big)_{L^2},$$

where $|D|=\sqrt{-\Delta}$. The orthogonality of cores (e.g. $\lim_{n\to\infty}|x_{n,j}-x_{n,k}|=+\infty$) implies in particular that the term $|D|^s\,V^k(x+(x_{n,j}-x_{n,k}))$ weakly converges to 0 in L^2 and thus (notice that $|D|^s\,V^j_\eta(x)$

belongs to L^2, by hypothesis)

$$\forall \eta > 0, \quad \forall (j,k) \in \mathcal{J}_1 \times \mathcal{J}_1, \quad \lim_{n\to\infty} \int_{\mathbb{R}^3} |D|^s V^j(x)\, |D|^s V^k(x + (x_{n,j} - x_{n,k}))\, dx = 0,$$

which ends up the proof of statement (30). Concerning statement (32), the proof is similar. Let $\varepsilon > 0$. As for any $T \leqslant \widetilde{T} - \varepsilon$, $NS(V_\eta^j)$ belongs to the space $X_T^s \stackrel{\text{def}}{=} \mathcal{C}_T(\dot{H}^s) \cap L_T^2(\dot{H}^{s+1})$. In particular, the map $t \in [0, \widetilde{T} - \varepsilon] \mapsto NS(V^j)(t, \cdot)$ belongs to $\dot{H}^s$. Previous computations hold and, by vertue of translation invariance of the $\dot{H}^s$-norm, we get for any $t < \widetilde{T}$ and $\eta > 0$,

(33)
$$\begin{aligned}\Big\| \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t, \cdot - x_{n,j}) \Big\|_{\dot{H}^s}^2 &= \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} \big\| NS(V^j)(t,\cdot)\big\|_{\dot{H}^s}^2 \\ &\quad + 2 \sum_{\substack{(j,k)\in\mathcal{J}_1\times\mathcal{J}_1\\ j\neq k}} \Big(|D|^s NS(V^j)(t, \cdot - x_{n,j}) \,\big|\, |D|^s NS(V^k)(t, \cdot - x_{n,k}) \Big)_{L^2}.\end{aligned}$$

Then, for any t in $[0, \widetilde{T} - \varepsilon]$, we get

$$\Big\| \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t, \cdot - x_{n,j}) \Big\|_{L_T^\infty(\dot{H}^s)}^2 \leqslant \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} \big\| NS(V^j)(t,\cdot)\big\|_{L_T^\infty(\dot{H}^s)}^2 + 2 \sum_{\substack{(j,k)\in\mathcal{J}_1\times\mathcal{J}_1\\ j\neq k}} \Gamma_{\varepsilon,n}^{s,j,k},$$

where $\Gamma_{\varepsilon,n}^{s,j,k}$ is defined by

$$(34)\qquad \begin{aligned}\Gamma_{\varepsilon,n}^{s,j,k} &\stackrel{\text{def}}{=} \sup_{t\in[0,\widetilde{T}-\varepsilon]} \Big(|D|^s NS(V^j)(t, \cdot - x_{n,j}) \,\big|\, |D|^s NS(V^k)(t, \cdot - x_{n,k}) \Big)_{L^2} \\ &= \sup_{t\in[0,\widetilde{T}-\varepsilon]} \int_{\mathbb{R}^3} |D|^s NS(V^j)(t,\cdot)\, |D|^s NS(V^k)(t, \cdot + (x_{n,j} - x_{n,k}))\, dx.\end{aligned}$$

The map $\psi : t \in [0, \widetilde{T} - \varepsilon] \mapsto |D|^s NS(V^j)(t,\cdot)\, |D|^s NS(V^k)(t, \cdot + (x_{n,j} - x_{n,k}))$ is continuous on the compact $[0, \widetilde{T} - \varepsilon]$, with value in $L^1(\mathbb{R}^3)$. Thus, $\psi([0, \widetilde{T} - \varepsilon])$ is precompact in the Lebesgue space $L^1(\mathbb{R}^3)$ and thus can be covered by a finite open ball with an arbitrarily radius $\alpha > 0$. Let α be a positive radius. There exists an integer N, such that for any $t \in [0, \widetilde{T} - \varepsilon]$, $\psi(t)$ belongs to $\displaystyle\bigcup_{l=1}^{N} \mathcal{B}\big(\psi(t_l), \alpha\big)$.

Thus, for any t belonging to the compact $[0, \widetilde{T} - \varepsilon]$, there exists a time t_l such that

$$(35)\qquad \|\psi(t)\|_{L^1(\mathbb{R}^3)} \leqslant \alpha + \|\psi(t_l)\|_{L^1(\mathbb{R}^3)}.$$

By vertue of the simple fact $\displaystyle\int f \leqslant \int |f|$, we infer that

$$\begin{aligned}\Gamma_{\varepsilon,n}^{s,j,k} &\leqslant \alpha + \|\psi(t_l)\|_{L^1(\mathbb{R}^3)} \\ &= \alpha + \int_{\mathbb{R}^3} \Big| |D|^s NS(V^j)(t_l,\cdot)\, |D|^s NS(V^k)(t_l, \cdot + (x_{n,j} - x_{n,k})) \Big|\, dx.\end{aligned}$$

Now, in order to conclude, we notice that Lebesgue theorem combining with the orthogonality property of cores imply that the right-hand-side tends to 0, when n tends to $+\infty$ (since we can choose α arbitrarily small) and thus, we get

$$\forall \eta > 0, \quad \forall (j,k) \in \mathcal{J}_1 \times \mathcal{J}_1, \quad \lim_{n\to\infty} \sup_{t\in[0,\widetilde{T}-\varepsilon]} \Big(|D|^s NS(V^j)(t, \cdot - x_{n,j}) \,\big|\, |D|^s NS(V^k)(t, \cdot - x_{n,k}) \Big)_{L^2} = 0.$$

Therefore, we have proved for any $T < \widetilde{T}$ and $\eta > 0$,

$$\Big\| \sum_{\substack{j \in \mathcal{J}_1 \\ j \leqslant J}} NS(V^j)(t, \cdot - x_{n,j}) \Big\|^2_{L^\infty_T(\dot{H}^s)} \leqslant \sum_{\substack{j \in \mathcal{J}_1 \\ j \leqslant J}} \big\| NS(V^j)(t, \cdot) \big\|^2_{L^\infty_T(\dot{H}^s)} + \circ(1), \quad \text{when} \quad n \to +\infty.$$

Concerning the $L^2_T(\dot{H}^{s+1})$-norm, we write estimate (33) in $\dot{H}^{s+1}$-norm. Then, the $L^2_T(\dot{H}^{s+1})$-norm of crossed terms tends to 0, thanks to Lebesgue theorem and orthogonality of cores. Details are left to the reader. Finally, we get (32)

$$\Big\| \sum_{\substack{j \in \mathcal{J}_1 \\ j \leqslant J}} NS(V^j)(t, \cdot - x_{n,j}) \Big\|^2_{X^s_T} \leqslant \sum_{\substack{j \in \mathcal{J}_1 \\ j \leqslant J}} \big\| NS(V^j)(t, \cdot) \big\|^2_{X^s_T} + \circ(1), \quad \text{when} \quad n \to +\infty.$$

In order to complete the proof of Proposition 4.1, we have to prove that the term $\sum_{\substack{j \in \mathcal{J}_1 \\ j \leqslant J}} \big\| NS(V^j)(t, \cdot) \big\|^2_{X^s_T}$ is bounded, *uniformly* in $\mathcal{J}_1$ (and thus in J) and η. This will result from Remark 1.1 and the orthogonality of $\dot{H}^s$-norm (9) in profile theorem. Indeed, by vertue of profile decomposition of the bounded sequence $(u_{0,n})_{n \geqslant 0}$ in the Sobolev space $\dot{H}^s$, we know that $\sum_{\substack{j \in \mathcal{J}_1 \\ j \leqslant J}} \|V^j\|^2_{\dot{H}^s}$ is bounded. It means that

$$\forall \varepsilon > 0, \ \exists \mathcal{J}_1^* \subset \mathcal{J}_1, \ \text{ with } \ |\mathcal{J}_1^*| < \infty \ \forall \ j \in \mathcal{J}_1 \setminus \mathcal{J}_1^*, \ \ \|V^j\|_{\dot{H}^s} \leqslant \varepsilon.$$

By vertue of Remark 1.1, we infer that for any j belonging to $\mathcal{J}_1 \setminus \mathcal{J}_1^*$, the Navier-Stokes solutions $NS(V^j)$ associated with such profiles V^j satisfy $\big\| NS(V^j)(t, \cdot) \big\|_{X^s_T} \leqslant 2 \big\| V^j \big\|_{\dot{H}^s}$. Therefore, we infer that

$$\begin{aligned} \sum_{\substack{j \in \mathcal{J}_1 \\ j \leqslant J}} \big\| NS(V^j)(t, \cdot) \big\|^2_{X^s_T} &\leqslant \sum_{\substack{j \in \mathcal{J}_1^* \\ j \leqslant J}} \big\| NS(V^j)(t, \cdot) \big\|^2_{X^s_T} \ + \ 4 \sum_{j \in \mathcal{J}_1 \setminus \mathcal{J}_1^*} \|V^j\|^2_{\dot{H}^s} \\ &\leqslant \sum_{\substack{j \in \mathcal{J}_1^* \\ j \leqslant J}} \big\| NS(V^j)(t, \cdot) \big\|^2_{X^s_T} \ + \ 4 \sum_{\substack{j \in \mathcal{J}_1 \\ j \leqslant J}} \|V^j\|^2_{\dot{H}^s} \\ &\leqslant \sum_{\substack{j \in \mathcal{J}_1^* \\ j \leqslant J}} \big\| NS(V^j)(t, \cdot) \big\|^2_{X^s_T} \ + \ 4 \limsup_{n \to +\infty} \|u_{0,n}\|^2_{\dot{H}^s}. \end{aligned} \tag{36}$$

As we are not so close to the blow up time (since $T < \inf_{j \in \mathcal{J}_1} T_*(V^j)$), the term $\sum_{\substack{j \in \mathcal{J}_1^* \\ j \leqslant J}} \big\| NS(V^j)(t, \cdot) \big\|^2_{X^s_T}$ is bounded, uniformly in $\mathcal{J}_1$ (since $\mathcal{J}_1^*$ is finite and depends only on the sequence of profiles V^j). Thus, the proof of Proposition 4.1 is complete.

Proof of Proposition 4.2. In order to prove the smallness result on the forcing term, we shall need to use the regularization process mentionned in the tool box of the previous section. Let us recall that we get an approximation of the Navier-Stokes solution associated with such a data, under the form of

$$NS(u_{0,n}) = U_n^{\text{app,J}} + R_n^J, \quad \text{where}$$

$$U_n^{\text{app,J}}(t, \cdot) \stackrel{\text{def}}{=} \sum_{\substack{j \in \mathcal{J}_1 \\ j \leqslant J}} NS(V^j)(t, \cdot - x_{n,j}) + e^{t\Delta} \Big(\sum_{\substack{j \in \mathcal{J}_1^c \\ j \leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j}, x_{n,j}} V^j(\cdot) + \psi_n^J(\cdot) \Big), \tag{37}$$

As already mentionned in (20), profiles are sorted with respect of the size of scales. Moreover, we cut off frequencies of profiles with small and big scales and therefore, decomposition (37) can be rewritten as follows

$$U_n^{\mathrm{app,J}}(t,\cdot\,) \stackrel{\mathrm{def}}{=} \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot - x_{n,j}) + e^{t\Delta}\Big(U^0_{n,\eta} + U^\infty_{n,\eta} \,+\, \psi^J_{n,\eta}(\cdot\,)\Big), \tag{38}$$

$$\begin{aligned} &\text{with}\quad U^0_{n,\eta} \stackrel{\mathrm{def}}{=} \sum_{\substack{j\in\mathcal{J}_0\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_\eta \quad ; \quad U^\infty_{n,\eta} \stackrel{\mathrm{def}}{=} \sum_{\substack{j\in\mathcal{J}_\infty\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_\eta \\ &\text{and}\quad \psi^J_{n,\eta} \stackrel{\mathrm{def}}{=} \sum_{\substack{j\in\mathcal{J}_0\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_{c\eta} \,+\, \sum_{\substack{j\in\mathcal{J}_\infty\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_{c\eta} \,+\, \psi^J_n. \end{aligned} \tag{39}$$

Let us point out that the main point is that, by vertue of Proposition 3.4, the terms $U^0_{n,\eta}$ and $U^\infty_{n,\eta}$ are small in the sense that, for any $\delta>0$, for any $\eta>0$, $\lim\limits_{n\to+\infty} ||U^0_{n,\eta}||_{\dot H^{s-\delta}} = 0$ and $\lim\limits_{n\to+\infty} ||U^\infty_{n,\eta}||_{\dot H^{s+\delta}} = 0$. We recall a basic property due to divergence free condition: for any vector field u, smooth enough and divergence-free,

$$u\cdot\nabla v = \mathrm{div}(u\otimes v)\cdot \tag{40}$$

The property (40) provides us another expression of the exterior force term F^J_n

$$F^J_n = I^{J,1}_{n,\eta} + I^{J,2}_{n,\eta} + I^{J,3}_n. \tag{41}$$

where

$$\begin{aligned} I^{J,1}_{n,\eta} &= \mathrm{div}\bigg(\Big(2\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot - x_{n,j}) + e^{t\Delta}\Big(U^0_{n,\eta} + U^\infty_{n,\eta} \,+\, \psi^J_{n,\eta}(x)\Big)\Big) \otimes e^{t\Delta}\Big(U^0_{n,\eta} + U^\infty_{n,\eta}\Big)\bigg). \\ I^{J,2}_{n,\eta} &= \mathrm{div}\bigg(\Big(2\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot - x_{n,j}) + e^{t\Delta}\Big(U^0_{n,\eta} + U^\infty_{n,\eta} \,+\, \psi^J_{n,\eta}(x)\Big)\Big) \otimes e^{t\Delta}\psi^J_{n,\eta}\bigg). \\ I^{J,3}_n &= F^{J,1}_n = \sum_{0\leqslant j,k\leqslant \mathcal{J}_1; j\neq k} NS(V^j(t,\cdot - x_{n,j}))\cdot \nabla NS(V^k)(t,\cdot - x_{n,k}). \end{aligned} \tag{42}$$

Concerning $I^{J,1}_{n,\eta}$, we apply (50) of Proposition 5.5, for any $\delta>0$, such that $s+\delta<\frac{3}{2}$,

$$\begin{aligned} \|I^{J,1}_{n,\eta}\|_{L^2_T(\dot H^{s-1})} &\leqslant C\,T^{\frac12(s-\frac12)}\,\Big(T^{\frac{-\delta}{2}}||U^0_{n,\eta}||_{\dot H^{s-\delta}} \,+\, T^{\frac{\delta}{2}}||U^\infty_{n,\eta}||_{\dot H^{s+\delta}}\Big) \\ &\qquad\times \Big\|2\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot - x_{n,j}) + e^{t\Delta}\Big(U^0_{n,\eta} + U^\infty_{n,\eta} + \psi^J_{n,\eta}\Big)\Big\|_{X^s_T} \\ &\leqslant C\,T^{\frac12(s-\frac12)}\,\Big(T^{\frac{-\delta}{2}}||U^0_{n,\eta}||_{\dot H^{s-\delta}} \,+\, T^{\frac{\delta}{2}}||U^\infty_{n,\eta}||_{\dot H^{s+\delta}}\Big) \\ &\qquad\times\bigg(2\,\|U_n^{\mathrm{app,J}}\|_{X^s_T} + \Big\|e^{t\Delta}\Big(U^0_{n,\eta} + U^\infty_{n,\eta} + \psi^J_{n,\eta}\Big)\Big\|_{X^s_T}\bigg). \end{aligned} \tag{43}$$

From (31), we infer that

$$\begin{aligned} \|I^{J,1}_{n,\eta}\|_{L^2_T(\dot H^{s-1})} &\leqslant C\,T^{\frac12(s-\frac12)}\,\Big(T^{\frac{-\delta}{2}}||U^0_{n,\eta}||_{\dot H^{s-\delta}} \,+\, T^{\frac{\delta}{2}}||U^\infty_{n,\eta}||_{\dot H^{s+\delta}}\Big) \\ &\qquad\times\Big(\|U_n^{\mathrm{app,J}}\|_{X^s_T} + \|u_{0,n}\|_{\dot H^s} + \circ(1)\Big)\,, \quad \text{when} \quad n\to+\infty. \end{aligned}$$

Propositions 3.4 and 4.1 implies that $I^{J,1}_{n,\eta}$ tends to 0 when n tends to infinity

$$\forall\varepsilon>0,\ \exists\tilde{n_1}(\varepsilon,J,\eta),\ \forall n\geqslant\tilde{n_1}(\varepsilon,J,\eta),\ \ \Big\|I^{J,1}_{n,\eta}\Big\|_{L^2_T(\dot H^{s-1})}\leqslant\varepsilon.$$

Concerning $I^{J,2}_{n,\eta}$, we apply the estimate (49) of Proposition 5.5

$$
\begin{aligned}
\|I^{J,2}_{n,\eta}\|_{L^2_T(\dot H^{s-1})} \leqslant C\,T^{\frac12(s-\frac12)}\,||\psi^J_{n,\eta}||_{\dot B^{s(1-\theta)}_{p_\theta,p_\theta}} \\
\times \Big\|2\sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} NS(V^j)(t,\cdot-x_{n,j}) + e^{t\Delta}\Big(U^0_{n,\eta}+U^\infty_{n,\eta}+\psi^J_{n,\eta}\Big)\Big\|_{X^s_T} \qquad (44)\\
\leqslant C\,T^{\frac12(s-\frac12)}\,||\psi^J_{n,\eta}||_{\dot B^{s(1-\theta)}_{p_\theta,p_\theta}}\ \big(\|U^{\mathrm{app,J}}_n\|_{X^s_T}+\|u_{0,n}\|_{\dot H^s}+\circ(1)\big)\,,\quad \text{when}\quad n\to+\infty.
\end{aligned}
$$

Thanks to Proposition 3.3, we infer

$$\forall\varepsilon>0,\ \exists\tilde J(\varepsilon),\ \forall J\geqslant\tilde J(\varepsilon),\ \exists\tilde\eta(J),\ \exists\tilde{n_2}(J),\forall\eta\geqslant\tilde\eta(J),\ \forall n\geqslant\tilde{n_2}(J),\ \ \|I^{J,2}_{n,\eta}\|_{L^2_T(\dot H^{s-1})}\leqslant\varepsilon.$$

Concerning $I^{J,3}_{n,\eta}$, the argument relies on the approximation Lemma 5.6 applied with with $\sigma=\frac{s}{2}+\frac34$, which proof is given in Appendix A. For the sake of simplicity, we note:

$$\varPhi^j = NS(V^j)\ \text{ and }\ \varPhi^k = NS(V^k).$$

Remark 4.1. As $\varPhi^j$ and $\varPhi^k$ belong to the space $L^\infty_T(\dot H^s)\cap L^2_T(\dot H^{s+1})$, an interpolation argument implies they belong to the space $L^4_T(\dot H^{\frac{s}{2}+\frac34})$. Indeed, we have

$$\|u\|_{\dot H^{\frac s2+\frac34}}\leqslant\|u\|^{\frac12(s+\frac12)}_{\dot H^s}\,\|u\|^{\frac12(\frac32-s)}_{\dot H^{s+1}}.$$

Then, by integrating in time, we deduce that

$$\|u\|^4_{L^4_T(\dot H^{\frac s2+\frac34})}\leqslant T^{s-\frac12}\,\|u\|^{2(s+\frac12)}_{L^\infty_T(\dot H^s)}\,\|u\|^{3-2s}_{L^2_T(\dot H^{s+1})}.$$

Thanks to the divergence-free condition, we have $\|\varPhi^j.\nabla\varPhi^k\|^2_{\dot H^{s-1}}=\|\varPhi^j\otimes\varPhi^k\|^2_{\dot H^s}$ and thus

$$
\begin{aligned}
\|\varPhi^j.\nabla\varPhi^k\|^2_{L^2_T(\dot H^{s-1})} &= \int_0^T\|\varPhi^j\otimes\varPhi^k\|^2_{\dot H^s}\\
&\leqslant\int_0^T\|(\varPhi^j-\varPhi^j_\varepsilon)\otimes\varPhi^k\|^2_{\dot H^s}+\int_0^T\|\varPhi^j_\varepsilon\otimes(\varPhi^k-\varPhi^k_\varepsilon)\|^2_{\dot H^s}+\int_0^T\|\varPhi^j_\varepsilon\otimes\varPhi^k_\varepsilon\|^2_{\dot H^s}.
\end{aligned}
$$

As $\dfrac s2+\dfrac34<\dfrac32$, a product rule in Sobolev spaces implies

$$\|u\,v\|_{\dot H^s}\leqslant C(s)\,\|u\|_{\dot H^{\frac s2+\frac34}}\,\|v\|_{\dot H^{\frac s2+\frac34}}. \tag{45}$$

Therefore, we infer that :

$$
\begin{aligned}
\|\varPhi^j.\nabla\varPhi^k\|^2_{L^2_T(\dot H^{s-1})}\lesssim&\int_0^T\|(\varPhi^j-\varPhi^j_\varepsilon)\|^2_{\dot H^{\frac s2+\frac34}}\|\varPhi^k\|^2_{\dot H^{\frac s2+\frac34}}+\int_0^T\|\varPhi^j_\varepsilon\|^2_{\dot H^{\frac s2+\frac34}}\|\varPhi^k-\varPhi^k_\varepsilon\|^2_{\dot H^{\frac s2+\frac34}}\\
&+\int_0^T\|\varPhi^j_\varepsilon\otimes\varPhi^k_\varepsilon\|^2_{\dot H^s}.
\end{aligned}
$$

Finally, Cauchy-Schwarz inequality and approximation Lemma 5.6 yield

$$\|\varPhi^j.\nabla\varPhi^k\|^2_{L^2_T(\dot H^{s-1})}\lesssim\varepsilon^2\|\varPhi^k\|^2_{L^4_T(\dot H^{\frac s2+\frac34})}+\varepsilon^2\|\varPhi^j\|^2_{L^4_T(\dot H^{\frac s2+\frac34})}+\|\varPhi^j_\varepsilon\otimes\varPhi^k_\varepsilon\|^2_{L^2_T(\dot H^s)}.$$

To conclude, we have to prove that $\|\varPhi^j_\varepsilon\otimes\varPhi^k_\varepsilon\|^2_{L^2_T(\dot H^s)}$ tends to 0, for ε small enough. This will come from the orthogonality of cores. By definition, $\varPhi^j_\varepsilon$ (resp. $\varPhi^k_\varepsilon$) is an approximation of $\varPhi^j$ (resp. $\varPhi^k$). Because of translations by cores, we define $\varPhi^{j,n}_\varepsilon(t,x-x_{n,j})$ (resp. $\varPhi^{k,n}_\varepsilon(t,x-x_{n,k})$) as an approximation of $\varPhi^j(t,x-x_{n,j})$ (resp. $\varPhi^k(t,x-x_{n,k})$). As $\varPhi^{j,n}_\varepsilon$ and $\varPhi^{k,n}_\varepsilon$ are compactly supported and concentrated around $x_{n,j}$ and $x_{n,k}$, the divergence of cores ($\lim\limits_{n\to+\infty}|x_{n,j}-x_{n,k}|=+\infty$) implies they are supported on disjointed compacts. Therefore, the term $\|\Phi^j_\varepsilon \otimes \Phi^k_\varepsilon\|^2_{L^2_T(\dot{H}^s)}$ converges to 0, for n large enough. In other words, we have

$$\forall \varepsilon > 0,\ \exists \tilde{n}(\varepsilon),\ \forall n \geqslant \tilde{n}(\varepsilon),\ \left\| NS(V^j(t, \cdot - x_{n,j}))\cdot \nabla NS(V^k)(t, \cdot - x_{n,k}) \right\|_{L^2_T(\dot{H}^{s-1})} \leqslant \frac{\varepsilon}{|\mathcal{J}_1|}.$$

Therefore, we infer that, $\forall \varepsilon > 0,\ \exists \tilde{n}(\varepsilon),\ \forall n \geqslant \tilde{n}(\varepsilon)$,

$$\left\| I^{J,3}_{n,\eta} \right\|_{L^2_T(\dot{H}^{s-1})} = \Big\| \sum_{\substack{0 \leqslant j,k \leqslant J\,;\, j \neq k \\ (j,k) \in \mathcal{J}_1^2}} NS(V^j(t, \cdot - x_{n,j}))\cdot \nabla NS(V^k)(t, \cdot - x_{n,k}) \Big\|_{L^2_T(\dot{H}^{s-1})} \leqslant \varepsilon. \tag{46}$$

This concludes the proof of Proposition 4.2.

5. Appendix A. Product and paraproduct estimates

In this section, we give some typical product estimates, in which splitting frequency allows for a much finer control of the product. The main tool is the homogeneous paradifferential calculus. For a detailed presentation of it, we refer the reader to [1], page 85. We recall two fundamental statements (see for instance Theorem 2.47 and 2.52 in [1]) about continuity of the homogeneous paraproduct operator T, and the remainder operator R. We shall constantly be using these two theorems in the sequel.

Theorem 5.1. *There exists a constant C such that for any real number s and any (p, r) in $[1, \infty]^2$, we have for any (u, v) in $L^\infty \times \dot{B}^s_{p,r}$,*

$$\|T_u v\|_{\dot{B}^s_{p,r}} \leqslant C^{1+|s|} \, \|u\|_{L^\infty} \, \|v\|_{\dot{B}^s_{p,r}}$$

Moreover, for any (s, t) in $\mathbb{R} \times]-\infty, 0[$, (p, r_1, r_2) in $[1, \infty]^3$, and (u, v) in $\dot{B}^t_{\infty,r_1} \times \dot{B}^s_{p,r_2}$, we have

$$\|T_u v\|_{\dot{B}^{s+t}_{p,r}} \leqslant \frac{C^{1+|s+t|}}{-t} \, \|u\|_{\dot{B}^t_{\infty,r_1}} \, \|v\|_{\dot{B}^s_{p,r_2}} \quad \text{with} \quad \frac{1}{r} \stackrel{\text{def}}{=} \min\left\{1, \frac{1}{r_1} + \frac{1}{r_2}\right\}.$$

Theorem 5.2. *A constant C exists which satisfies the following properties.*
Let (s_1, s_2) be in $\mathbb{R}^2$ and (p_1, p_2, r_1, r_2) in $[1, \infty]^4$. Let us assume that

$$\frac{1}{p} \stackrel{\text{def}}{=} \frac{1}{p_1} + \frac{1}{p_2} \leqslant 1 \quad \text{and} \quad \frac{1}{r} \stackrel{\text{def}}{=} \frac{1}{r_1} + \frac{1}{r_2} \leqslant 1.$$

If $s_1 + s_2$ is positive, then we have for any (u, v) in $\dot{B}^{s_1}_{p_1,r_1} \times \dot{B}^{s_2}_{p_2,r_2}$,

$$\|R(u, v)\|_{\dot{B}^{s_1+s_2}_{p,r}} \leqslant \frac{C^{1+|s_1+s_2|}}{s_1 + s_2} \|u\|_{\dot{B}^{s_1}_{p_1,r_1}} \, \|v\|_{\dot{B}^{s_2}_{p_2,r_2}}.$$

A lot of results of continuity may be deduced from the two above Theorems. For instance, we can state the Lemma below.

Lemma 5.3. *(Product rule in $\dot{H}^s$)*
Let u and v be two functions in $\dot{H}^s$ with $-\frac{3}{2} < s < \frac{3}{2}$, then

$$\|uv\|_{\dot{H}^s} \leqslant C(s) \left(\|u\|_{\dot{H}^s} \, \|v\|_{\dot{H}^{\frac{3}{2}}} + \|u\|_{\dot{H}^{\frac{3}{2}}} \, \|v\|_{\dot{H}^s} \right) \quad \text{and} \quad \|uv\|_{\dot{H}^s} \leqslant C(s) \, \|u\|_{L^\infty \cap \dot{H}^{\frac{3}{2}}} \, \|v\|_{\dot{H}^s}.$$

Proof. We have to estimate a product in Sobolev space thus, we shall use the paradifferential calculus. In particular, thanks to the Bony's paraproduct decomposition, we get

$$u\, v = T_u v + R(u, v) + T_v u.$$

The term $R(v, u)$ can be estimated in $\dot{H}^s$-norm easily, thanks to Theorem 5.2,

$$\|R(u, v)\|_{\dot{B}^{s+\frac{3}{2}}_{1,1}} \leqslant C \|u\|_{\dot{H}^s} \, \|v\|_{\dot{H}^{\frac{3}{2}}}.$$

Therefore, thanks to embeddings $\dot{B}^{s+\frac{3}{2}}_{1,1} \hookrightarrow \dot{B}^{s}_{2,1} \hookrightarrow \dot{B}^{s}_{2,2}$ and Remark 3.1, we infer that

$$\|R(u,v)\|_{\dot{H}^s} \leqslant C\|u\|_{\dot{H}^s}\|v\|_{\dot{H}^{\frac{3}{2}}}. \tag{47}$$

Concerning $T_v u$ and $T_u v$, we use once again estimates of Theorem 5.1, which gives

$$\|T_u v\|_{\dot{H}^s} \leqslant C(s)\, \|u\|_{\dot{B}^{s-\frac{3}{2}}_{\infty,\infty}}\, \|v\|_{\dot{B}^{\frac{3}{2}}_{2,2}}.$$

Because $s-\frac{3}{2}$ is negative, Bernstein's inequality and the classical embedding $\ell^2(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$ give

$$\dot{B}^{s}_{2,2} \hookrightarrow \dot{B}^{s-\frac{3}{2}}_{\infty,2} \hookrightarrow \dot{B}^{s-\frac{3}{2}}_{\infty,\infty}.$$

Therefore, we deduce that

$$\|T_u v\|_{\dot{H}^s} \leqslant C(s)\, \|u\|_{\dot{H}^s}\, \|v\|_{\dot{H}^{\frac{3}{2}}}.$$

Permuting the roles of u and v and using (47) gives the first part of the result. The second part of Lemma 5.3 is easy. By vertue of Theorem 5.1, we have

$$\|T_u v\|_{\dot{H}^s} \leqslant C(s)\, \|u\|_{L^\infty}\, \|v\|_{\dot{H}^s} \quad \text{and} \quad \|T_v u\|_{\dot{H}^s} \leqslant C(s)\, \|v\|_{\dot{H}^s}\, \|u\|_{\dot{H}^{\frac{3}{2}}}.$$

Moreover, it seems clear that, due to Theorem 5.2

$$\|R(u,v)\|_{\dot{B}^{s+\frac{3}{2}}_{1,1}} \leqslant C\|v\|_{\dot{H}^s}\, \|u\|_{\dot{H}^{\frac{3}{2}}}.$$

This leads to the proof of $\|uv\|_{\dot{H}^s} \leqslant\; C(s)\, \|u\|_{L^\infty \cap \dot{H}^{\frac{3}{2}}}\, \|v\|_{\dot{H}^s}$. □

Remark 5.1. Let us point out an interpolation inequality: by definition of s we have

$$\|u\|_{\dot{H}^{\frac{3}{2}}} \leqslant C\|u\|^{s-\frac{1}{2}}_{\dot{H}^s}\, \|u\|^{\frac{3}{2}-s}_{\dot{H}^{s+1}}. \tag{48}$$

Therefore, combining this with Lemma 5.3, we get the result following which will be a frequent use later on.

Corollary 5.4. *Let u and v be in $\dot{H}^s$ with $\dfrac{1}{2} < s < \dfrac{3}{2}$, then*

$$\|uv\|_{\dot{H}^s} \leqslant\; C(s)\Big(\|u\|_{\dot{H}^s}\, \|v\|^{s-\frac{1}{2}}_{\dot{H}^s}\, \|v\|^{\frac{3}{2}-s}_{\dot{H}^{s+1}} + \|u\|^{s-\frac{1}{2}}_{\dot{H}^s}\, \|u\|^{\frac{3}{2}-s}_{\dot{H}^{s+1}}\, \|v\|_{\dot{H}^s}\Big).$$

Proposition 5.5. *Let $0<\theta<1$. Under the interpolation relation $\dfrac{1}{p_\theta} = \dfrac{\theta}{p} + \dfrac{1-\theta}{2}$,*

$$\|u \otimes e^{t\Delta} r_0\|_{L^2_T(\dot{H}^s)} \leqslant C\; T^{\frac{1}{2}(s-\frac{1}{2})}\|u\|_{X^s_T}||r_0||_{\dot{B}^{s(1-\theta)}_{p_\theta,p_\theta}}. \tag{49}$$

$$\text{For any } \frac{1}{2} < \alpha < \frac{3}{2},\; \|u \,\otimes\, e^{t\Delta} r_0\|_{L^2_T(\dot{H}^s)} \leqslant C\; T^{\frac{1}{2}(\alpha-\frac{1}{2})}\|u\|_{X^s_T}||r_0||_{\dot{H}^\alpha}. \tag{50}$$

Proof. Let us start by proving the first inequality. Bony's paraproduct decomposition implies

$$u \otimes e^{t\Delta} r_0 = T_{e^{t\Delta} r_0} u + R(e^{t\Delta} r_0, u) + T_u(e^{t\Delta} r_0).$$

The first two terms can be estimated in $\dot{H}^s$-norm easily. Thanks to Theorem 5.1, we have

$$\|T_{e^{t\Delta} r_0}(u)\|_{\dot{H}^s=\dot{B}^s_{2,2}} \leqslant C\|e^{t\Delta} r_0\|_{\dot{B}^{s-\frac{3}{2}}_{\infty,\infty}}\, \|u\|_{\dot{B}^{\frac{3}{2}}_{2,2}}.$$

Let us recall that $\dfrac{1}{p_\theta}$ is defined by $\dfrac{1}{p_\theta} = \dfrac{\theta}{p} + \dfrac{1-\theta}{2}$, for any θ in $]0,1[$.

A classical result due to Bernstein's inequality gives the following embedding $\dot{B}^{s(1-\theta)}_{p_\theta,\infty} \hookrightarrow \dot{B}^{s-\frac{3}{2}}_{\infty,\infty}$. Therefore we infer that

$$\|T_{e^{t\Delta} r_0}(u)\|_{L^2_T(\dot{H}^s)} \lesssim \|e^{t\Delta} r_0\|_{L^\infty_T(\dot{B}^{s(1-\theta)}_{p_\theta,\infty})}\|u\|_{L^2_T(\dot{H}^{\frac{3}{2}})}.$$

On the one hand, thanks to the hypothesis $\frac{1}{2} < s < \frac{3}{2}$, we recover the Navier-Stokes solution u in X^s_T-norm by an interpolation argument. As $\|u\|_{\dot H^{\frac32}} \lesssim \|u\|_{\dot H^s}^{s-\frac12} \|u\|_{\dot H^{s+1}}^{\frac32 - s}$, we get

$$\begin{aligned}\|u\|^2_{L^2_T(\dot H^{\frac32})} = \int_0^T \|u\|^{2s-1}_{\dot H^s}\|u\|^{3-2s}_{\dot H^{s+1}}dt &\lesssim \|u\|^{2s-1}_{L^\infty_T(\dot H^s)}\ T^{s-\frac12}\ \|u\|^{3-2s}_{L^2_T(\dot H^{s+1})}\\ &\lesssim T^{s-\frac12}\ \|u\|^2_{X^s_T}.\end{aligned} \tag{51}$$

On the other hand, the simple embedding $\ell^{p_\theta}(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$ implies

$$\|e^{t\Delta}r_0\|_{L^\infty_T(\dot B^{s(1-\theta)}_{p_\theta,\infty})} \leqslant \|r_0\|_{\dot B^{s(1-\theta)}_{p_\theta,\infty}} \leqslant \|r_0\|_{\dot B^{s(1-\theta)}_{p_\theta,p_\theta}}.$$

Finally, we have proved the proposition for the first two terms

$$\|R(e^{t\Delta}r_0,u)\|_{L^2_T(\dot H^s)} + \|T_{e^{t\Delta}r_0}(u)\|_{L^2_T(\dot H^s)} \lesssim T^{\frac12(s-\frac12)}\ \|r_0\|_{\dot B^{s(1-\theta)}_{p_\theta,p_\theta}}\|u\|_{X^s_T}.$$

The last term $T_u(e^{t\Delta}r_0)$ is more delicate. Note that, here, as we work locally in time, low frequencies do not play a major role, unlike high frequencies. As a result, we have to handle low and high frequencies separately. It is natural to split them according to their size: either the frequencies are low (in the sense that $\sqrt{T}2^j \leqslant C$) or the frequenties are high (in the sense that $\sqrt{T}2^j \geqslant C$).
Firstly, let us observe that

$$\|T_u(e^{t\Delta}r_0)\|_{L^2_T(\dot H^s)} = \|T_u(e^{t\Delta}r_0)\|_{L^2_T(\dot B^s_{2,2})} = \Big(2^{js}\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}\Big)_{\ell^2(\mathbb{Z})}.$$

We split, according to low and high frequencies

$$\begin{aligned}\|T_u(e^{t\Delta}r_0)\|_{L^2_T(\dot H^s)} \leqslant &\Big(2^{js}\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}1_{\{\sqrt{T}2^j\leqslant C\}}\Big)_{\ell^2(\mathbb{Z})}\\ &+\Big(2^{js}\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}1_{\{\sqrt{T}2^j\geqslant C\}}\Big)_{\ell^2(\mathbb{Z})}.\end{aligned} \tag{52}$$

A classical result in Littlewood Paley theory gives the following estimates

$$\dot\Delta_j T_u(e^{t\Delta}r_0) = \sum_{|j-j'|\leqslant 4} \dot S_{j'-1}u\ \dot\Delta_{j'}(e^{t\Delta}r_0).$$

Therefore, Hölder's inequality yields

$$\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2} \leqslant \sum_{|j-j'|\leqslant 4} \|\dot S_{j'-1}u\|_{L^{q_\theta}}\|\dot\Delta_{j'}(e^{t\Delta}r_0)\|_{L^{p_\theta}} \quad \text{with} \quad \frac12 = \frac{1}{p_\theta}+\frac{1}{q_\theta}.$$

In particular, Bernstein's inequality implies

$$\begin{aligned}\|\dot S_{j'-1}u\|_{L^{q_\theta}} &\leqslant \sum_{j''=-\infty}^{j'-2}\|\dot\Delta_{j''}u\|_{L^{q_\theta}}\\ &\lesssim \sum_{j''=-\infty}^{j'-2} 2^{3j''(\frac12-\frac{1}{q_\theta})}\|\dot\Delta_{j''}u\|_{L^2} = \sum_{j''=-\infty}^{j'-2} 2^{j''(\frac{3}{p_\theta}-s)}2^{j''s}\|\dot\Delta_{j''}u\|_{L^2}.\end{aligned}$$

Applying Young's inequality, we infer there exists for any t, a sequence $(c_j(t))_{j\in\mathbb{Z}}$ belonging to the sphere of $\ell^{q_\theta}(\mathbb{Z})$, such that

$$\|\dot S_{j'-1}u\|_{L^{q_\theta}} \leqslant C\, c_{j'}(t)\ 2^{j'(\frac{3}{p_\theta}-s)}\ \|u(t)\|_{\dot B^s_{2,q_\theta}}.$$

As $q_\theta \geqslant 2$, $\ell^2(\mathbb{Z})$ is included in $\ell^{q_\theta}(\mathbb{Z})$, which implies that

$$\|\dot S_{j'-1}u\|_{L^{q_\theta}} \leqslant C\, c_{j'}(t)\ 2^{j'(\frac{3}{p_\theta}-s)}\ \|u(t)\|_{\dot B^s_{2,2}}.$$

Therefore, we have

$$\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2} \lesssim \sum_{|j-j'|\leqslant 4} c_{j'}(t) 2^{j'(\frac{3}{p_\theta}-2s+\theta\, s)} \|u(t)\|_{\dot B^s_{2,2}}\ 2^{sj'(1-\theta)} \|\dot\Delta_{j'}(e^{t\Delta}r_0)\|_{L^{p_\theta}}.$$

As j and j' are equivalent, we can write

$$\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2} \lesssim c_j(t) 2^{j(\frac{3}{p_\theta}-2s+\theta\, s)} \|u(t)\|_{\dot B^s_{2,2}}\ 2^{js(1-\theta)} \|\dot\Delta_j(e^{t\Delta}r_0)\|_{L^{p_\theta}}. \tag{53}$$

On the other hand, we have (see for instance Lemma 2.4 of [1])

$$\|\dot\Delta_{j'}(e^{t\Delta}r_0)\|_{L^{p_\theta}} \lesssim e^{-t2^{2j'}} \|\dot\Delta_{j'}r_0\|_{L^{p_\theta}}. \tag{54}$$

As $e^{-t2^{2j'}} \leqslant 1$, integration in time yields

$$\|\dot\Delta_{j'}(e^{t\Delta}r_0)\|_{L^{p_\theta}_T(L^{p_\theta})} \lesssim T^{\frac{1}{p_\theta}} \|\dot\Delta_{j'}r_0\|_{L^{p_\theta}}.$$

Above result combining with Hölder's inequality in time imply

$$2^{js}\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)} \lesssim 2^{j(\frac{3}{p_\theta}-s+\theta\, s)} \|u(t)\|_{L^\infty_T(\dot H^s)} \|c_j(t)\|_{L^{q_\theta}_T}\ T^{\frac{1}{p_\theta}}\ 2^{js(1-\theta)}\ \|\dot\Delta_j r_0\|_{L^{p_\theta}}.$$

Therefore, as far as the low frequencies are concerned ($\sqrt{T}2^j \leqslant C$), we have

$$\begin{aligned}2^{js}\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)} 1_{\{\sqrt{T}2^j\leqslant C\}} &\lesssim T^{-\frac12(\frac{3}{p_\theta}-s+\theta\, s)} \|u(t)\|_{L^\infty_T(\dot H^s)} \|c_j(t)\|_{L^{q_\theta}_T}\\ &\quad\times T^{\frac{1}{p_\theta}}\ 2^{js(1-\theta)}\ \|\dot\Delta_j r_0\|_{L^{p_\theta}}.\end{aligned}$$

Applying Hölder's inequality for the $\ell^2(\mathbb{Z})$-norm, we have

$$\begin{aligned}\Big(2^{js}\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)} 1_{\{\sqrt{T}2^j\leqslant C\}}\Big)_{\ell^2(\mathbb{Z})} &\leqslant T^{-\frac12(\frac{3}{p_\theta}-s+\theta\, s)+\frac{1}{p_\theta}}\ \|u(t)\|_{L^\infty_T(\dot H^s)}\\ &\quad\times \Big(\|c_j(t)\|_{L^{q_\theta}_T}\Big)_{\ell^{q_\theta}(\mathbb{Z})} \|r_0\|_{\dot B^{s(1-\theta)}_{p_\theta,p_\theta}}.\end{aligned}$$

Clearly, we have $\Big(\|c_j(t)\|_{L^{q_\theta}_T}\Big)_{\ell^{q_\theta}(\mathbb{Z})} \leqslant T^{\frac{1}{q_\theta}}$. Besides, we have

$$\begin{aligned}-\frac12\left(\frac{3}{p_\theta}-s+\theta\, s\right)+\frac{1}{p_\theta}+\frac{1}{q_\theta} &= -\frac12\left(\frac{3}{p_\theta}-s+\theta\, s-1\right) \;=\; -\frac12\left(\frac{3\theta}{p}+\frac{3(1-\theta)}{2}-s+\theta\, s-1\right)\\ &= -\frac12\left(\frac{3\theta}{2}-\theta\, s+\frac{3(1-\theta)}{2}-s+\theta\, s-1\right) \;=\; \frac12\left(s-\frac12\right)\cdot\end{aligned}$$

As a result, we infer that

$$\Big(2^{js}\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)} 1_{\{\sqrt{T}2^j\leqslant C\}}\Big)_{\ell^2(\mathbb{Z})} \leqslant T^{\frac12(s-\frac12)}\ \|u(t)\|_{L^\infty_T(\dot H^s)} \|r_0\|_{\dot B^{s(1-\theta)}_{p_\theta,p_\theta}}.$$

This completes the proof in the case of low frequencies. For the high frequencies, we need to use the smoothing effect of the heat flow. Thanks to (54), we infer

$$\|\dot\Delta_{j'}(e^{t\Delta}r_0)\|_{L^{p_\theta}_T(L^{p_\theta})} \lesssim 2^{\frac{-2j'}{p_\theta}} \|\dot\Delta_{j'}r_0\|_{L^{p_\theta}}. \tag{55}$$

We write an estimate for $2^{js}\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)} 1_{\{\sqrt{T}2^j\geqslant C\}}$. We come back to (53), we integrate in time, applying Hölder's inequality

$$\begin{aligned}2^{js}\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)} &\lesssim 2^{j(\frac{1}{p_\theta}-s+\theta\, s)}\ \|u(t)\|_{L^\infty_T(\dot H^s)}\ \|c_j(t)\|_{L^{q_\theta}_T}\\ &\quad\times 2^{j\left((1-\theta)s+\frac{2}{p_\theta}\right)}\ \|\dot\Delta_j(e^{t\Delta}r_0)\|_{L^{p_\theta}_T(L^{p_\theta})}.\end{aligned} \tag{56}$$

High frequencies hypothesis implies

$$2^{js}\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}1_{\{\sqrt{T}2^j\geqslant C\}} \lesssim T^{-\frac{1}{2}(\frac{1}{p_\theta}-s+\theta\, s)}\|u(t)\|_{L^\infty_T(\dot H^s)}\|c_j(t)\|_{L^{q_\theta}_T}$$
$$\times\, 2^{j((1-\theta)s+\frac{2}{p_\theta})}\ \|\dot\Delta_j(e^{t\Delta}r_0)\|_{L^{p_\theta}_T(L^{p_\theta})}.$$

Thanks to (55), we infer

$$2^{js}\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}1_{\{\sqrt{T}2^j\geqslant C\}} \lesssim T^{-\frac{1}{2}(\frac{1}{p_\theta}-s+\theta\, s)}\|u(t)\|_{L^\infty_T(\dot H^s)}\|c_j(t)\|_{L^{q_\theta}_T}$$
$$\times\, 2^{j((1-\theta)s)}\ \|\dot\Delta_j r_0\|_{L^{p_\theta}}. \tag{57}$$

Once again, we apply Hölder's inequality for the $\ell^2(\mathbb{Z})$-norm and we have

$$\Big(2^{js}\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}1_{\{\sqrt{T}2^j\geqslant C\}}\Big)_{\ell^2(\mathbb{Z})} \lesssim T^{-\frac{1}{2}(\frac{1}{p_\theta}-s+\theta\, s)+\frac{1}{q_\theta}}\|u(t)\|_{L^\infty_T(\dot H^s)}\|r_0\|_{L^{p_\theta}_T(\dot B^{(1-\theta)s}_{p_\theta,p_\theta})}.$$

Then, the simple computation $-\dfrac{1}{2}\Big(\dfrac{1}{p_\theta}-s+\theta\, s\Big)+\dfrac{1}{q_\theta}=\dfrac{1}{2}\Big(s-\dfrac{1}{2}\Big)$ implies

$$\Big(2^{j}\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}1_{\{\sqrt{T}2^j\geqslant C\}}\Big)_{\ell^2(\mathbb{Z})} \lesssim T^{\frac{1}{2}(s-\frac{1}{2})}\ \|u(t)\|_{L^\infty_T(\dot H^s)}\|r_0\|_{\dot B^{s(1-\theta)}_{p_\theta,p_\theta}}.$$

This ends up the proof for the case of high frequencies and therefore the first inequality of the proposition is proved.

Now, let us prove the second inequality, which proof is very close to the previous one. We gives only outlines. Thanks to Bony's decomposition, we have

$$u\otimes e^{t\Delta}r_0 = T_{e^{t\Delta}r_0}u + R(e^{t\Delta}r_0,u)+T_u(e^{t\Delta}r_0).$$

The first two terms can be estimated in $\dot H^s$-norm easily, thanks to mapping of paraproduct in the Besov spaces (cf Theorem 5.1)

$$\|T_{e^{t\Delta}r_0}u\|_{\dot B^s_{2,2}} \leqslant C\|e^{t\Delta}r_0\|_{\dot B^{\alpha-\frac{3}{2}}_{\infty,\infty}}\|u\|_{\dot B^{s+\frac{3}{2}-\alpha}_{2,2}}.$$

On the one hand, Bernstein's Lemma and obvious embedding $\ell^2(\mathbb{Z})\subset\ell^\infty(\mathbb{Z})$ ensure that

$$\dot B^\alpha_{2,2}\hookrightarrow \dot B^{\alpha-\frac{3}{2}}_{\infty,2}\hookrightarrow \dot B^{\alpha-\frac{3}{2}}_{\infty,\infty}\quad\text{and thus}\quad \|e^{t\Delta}r_0\|_{\dot B^{\alpha-\frac{3}{2}}_{\infty,\infty}}\lesssim \|e^{t\Delta}r_0\|_{\dot B^\alpha_{2,2}}.$$

On the other hand, as $s\leqslant s+\frac{3}{2}-\alpha\leqslant s+1$, u belongs to $\dot B^{s+\frac{3}{2}-\alpha}_{2,2}$. Interpolation argument yields

$$\|u\|_{\dot B^{s+\frac{3}{2}-\alpha}_{2,2}}\leqslant C\,\|u\|_{\dot H^{s+\frac{3}{2}-\alpha}}\leqslant \|u\|^{\alpha-\frac{1}{2}}_{\dot H^s}\ \|u\|^{\frac{3}{2}-\alpha}_{\dot H^{s+1}}.$$

By integration in time and thanks to Hölder's inequality, we have

$$\|u\|^2_{L^2_T(\dot H^{s+\frac{3}{2}-\alpha})}\leqslant \int_0^T \|u(t,\cdot)\|^{2\alpha-1}_{\dot H^s}\ \|u(t,\cdot)\|^{3-2\alpha}_{\dot H^{s+1}}\,dt$$
$$\leqslant T^{\alpha-\frac{1}{2}}\ \|u\|^{2\alpha-1}_{L^\infty_T(\dot H^s)}\ \|u\|^{3-2\alpha}_{L^2_T(\dot H^{s+1})}.$$

Finally, we get

$$\|u\|_{L^2_T(\dot H^{s+\frac{3}{2}-\alpha})}\leqslant T^{\frac{1}{2}(\alpha-\frac{1}{2})}\ \|u\|_{X^s_T}.$$

Therefore, we deduce an estimate of the term $\|T_{e^{t\Delta}r_0}(u)\|_{L^2_T(\dot H^s)}$ and $\|R(e^{t\Delta}r_0,u)\|_{L^2_T(\dot H^s)}$.

$$\|T_{e^{t\Delta}r_0}u\|_{L^2_T(\dot H^s)}\leqslant T^{\frac{1}{2}(\alpha-\frac{1}{2})}\ \|u\|_{X^s_T}\|r_0\|_{\dot H^\alpha}.$$

$$\|R(e^{t\Delta}r_0,u)\|_{L^2_T(\dot H^s)}\leqslant T^{\frac{1}{2}(\alpha-\frac{1}{2})}\ \|u\|_{X^s_T}\|r_0\|_{\dot H^\alpha}.$$

Now, in order to estimate the last term $\|T_u(e^{t\Delta}r_0)\|_{L^2_T(\dot H^s)}$, we shall need splitting, according low and high frequencies (e.g $\sqrt{T}\,2^j \leqslant 1$ or $\sqrt{T}\,2^j \geqslant 1$). That is exactly the same computations as in the proof of the first inequality of the proposition

$$\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2} \lesssim \sum_{|j-j'|\leqslant 4} \|\dot S_{j'-1}u\|_{L^p}\|\dot\Delta_{j'}(e^{t\Delta}r_0)\|_{L^{\frac{3}{s}}}.$$

Thanks to the property $\|\dot S_{j'-1}u\|_{L^p} \lesssim \|u\|_{L^p}$ and the equivalence between j and j', we get

$$\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2} \lesssim \|u\|_{L^p}\|\dot\Delta_j(e^{t\Delta}r_0)\|_{L^{\frac{3}{s}}}.$$

By vertue of Sobolev embedding and integration in time

$$2^{js}\,\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)} \leqslant 2^{js}\,\|u\|_{L^\infty_T(\dot H^s)}\;\|\dot\Delta_j(e^{t\Delta}r_0)\|_{L^2_T(L^{\frac{3}{s}})}.$$

Concerning low frequencies (e.g $\sqrt{T}\,2^j \leqslant 1$), we combine (54) with the rough boundary $e^{-t2^{2j}} \leqslant 1$ and we get

$$\begin{aligned}
2^{js}\,\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}\,1_{\{\sqrt{T}2^j\leqslant C\}} &\lesssim 2^{js}\,\|u\|_{L^\infty_T(\dot H^s)}\;\|\dot\Delta_j(r_0)\|_{L^2_T(L^{\frac{3}{s}})}\\
&\lesssim 2^{js}\,\|u\|_{L^\infty_T(\dot H^s)}\;2^{-j(\alpha+s-\frac{3}{2})}\;2^{j(\alpha+s-\frac{3}{2})}\;\|\dot\Delta_j(r_0)\|_{L^2_T(L^{\frac{3}{s}})}\\
&\lesssim 2^{-j(\alpha-\frac{3}{2})}\,\|u\|_{L^\infty_T(\dot H^s)}\;2^{j(\alpha+s-\frac{3}{2})}\;T^{\frac{1}{2}}\;\|\dot\Delta_j(r_0)\|_{L^{\frac{3}{s}}}.
\end{aligned}$$

Hypothesis of low frequencies implies

$$\Big(2^{js}\,\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}\,1_{\{\sqrt{T}2^j\leqslant C\}}\Big)_{\ell^2(\mathbb{Z})} \lesssim T^{\frac{1}{2}(\alpha-\frac{1}{2})}\;\|u\|_{L^\infty_T(\dot H^s)}\;\|r_0\|_{\dot B^{\alpha+s-\frac{3}{2}}_{\frac{3}{s},2}}. \tag{58}$$

As far as high frequencies are concerned (e.g $\sqrt{T}\,2^j \geqslant 1$), (54) combining with the integration of the term $e^{-t2^{2j}}$ on $[0,T]$, gives

$$\begin{aligned}
2^{js}\,\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}\,1_{\{\sqrt{T}2^j\geqslant C\}} &\lesssim 2^{js}\,\|u\|_{L^\infty_T(\dot H^s)}\;2^{-j}\|\dot\Delta_j(r_0)\|_{L^{\frac{3}{s}}}\\
&\lesssim 2^{j(s-1)}\,\|u\|_{L^\infty_T(\dot H^s)}\;2^{-j(\alpha+s-\frac{3}{2})}\;2^{j(\alpha+s-\frac{3}{2})}\;\|\dot\Delta_j(r_0)\|_{L^{\frac{3}{s}}}\\
&\lesssim 2^{-j(\alpha-\frac{1}{2})}\,\|u\|_{L^\infty_T(\dot H^s)}\;2^{j(\alpha+s-\frac{3}{2})}\;\|\dot\Delta_j(r_0)\|_{L^{\frac{3}{s}}}.
\end{aligned}$$

Hypothesis of high frequencies gives

$$\Big(2^{js}\,\|\dot\Delta_j T_u(e^{t\Delta}r_0)\|_{L^2_T(L^2)}\,1_{\{\sqrt{T}2^j\geqslant C\}}\Big)_{\ell^2(\mathbb{Z})} \lesssim T^{\frac{1}{2}(\alpha-\frac{1}{2})}\;\|u\|_{L^\infty_T(\dot H^s)}\;\|r_0\|_{\dot B^{\alpha+s-\frac{3}{2}}_{\frac{3}{s},2}}. \tag{59}$$

Combining (58) and (59) with the fact that $\dot B^{\alpha}_{2,2}$ is embedded in $\dot B^{\alpha+s-\frac{3}{2}}_{\frac{3}{s},2}$, we get finally

$$\|T_u(e^{t\Delta}r_0)\|_{L^2_T(\dot H^s)} \lesssim T^{\frac{1}{2}(\alpha-\frac{1}{2})}\;\|u\|_{L^\infty_T(\dot H^s)}\;\|r_0\|_{\dot H^\alpha}. \tag{60}$$

This completes the proof of the second inequality of the proposition. Now, let us state an approximation lemma. □

Lemma 5.6. *Let $0 < \sigma < \frac{3}{2}$ and $\varepsilon > 0$. Let a be an element of $L^4_T(\dot H^\sigma)$. Then, there exists a constant $C > 0$, there exists a family of compactly supported functions, a_ε, which satisfies for any positive T*

$$\lim_{\varepsilon\to 0}\|a-a_\varepsilon\|_{L^4_T(\dot H^\sigma)} = 0 \quad \textit{and} \tag{61}$$

$$\|a_\varepsilon\|_{L^4_T(\dot H^\sigma)} \leqslant C\,\|a\|_{L^4_T(\dot H^\sigma)}. \tag{62}$$

Proof. Let us introduce the approximation function a_ε defined by

$$a_\varepsilon = \chi(\varepsilon\cdot)a,$$

where χ is the usual fonction of $\mathcal{D}(\mathbb{R}^3)$ with value 1 near 0.
Let us start by proving (62). Due to product rule in Sobolev spaces recalled in Lemma 5.3, we have

$$\begin{aligned}\|a_\varepsilon\|_{\dot H^\sigma} &\leqslant C(\sigma)\,\|\chi(\varepsilon\cdot)\|_{L^\infty\cap\dot H^{\frac32}}\,\|a\|_{\dot H^\sigma}\\ &\leqslant C(\sigma)\,\|\chi\|_{L^\infty\cap\dot H^{\frac32}}\,\|a\|_{\dot H^\sigma}\\ &\leqslant C(\sigma)\,\|a\|_{\dot H^\sigma}\end{aligned} \tag{63}$$

Now we prove (61). In order to apply Lebesgue Theorem, we have to prove there exists a positive constant C, such that

$$\lim_{\varepsilon\to0}\|a_\varepsilon-a\|_{\dot H^\sigma}=0 \quad\text{and}\quad \|a_\varepsilon-a\|_{\dot H^\sigma}\leqslant C. \tag{64}$$

Let us notice that $\|a_\varepsilon-a\|_{\dot H^\sigma}$ is bounded, thanks to (63). Concerning the proof of $\lim\limits_{\varepsilon\to0}\|a_\varepsilon-a\|_{\dot H^\sigma}=0$, one way is to approach the function a by an truncated element A_η which Fourier transform is defined by $\widehat{A_\eta}(\xi)=\widehat{a}(\xi)1_{\{\eta\leqslant|\xi|\leqslant\frac1\eta\}}$. In this way, by vertue of Lebesgue Theorem, it seems clear that

$$\lim_{\eta\to0}\|A_\eta-a\|_{\dot H^\sigma}=0. \tag{65}$$

Therefore, we have

$$\begin{aligned}\|a_\varepsilon-a\|_{\dot H^\sigma} &= \|(1-\chi(\varepsilon\cdot))\,a\|_{\dot H^\sigma}\\ &\leqslant \|(1-\chi(\varepsilon\cdot))\,(a-A_\eta)\|_{\dot H^\sigma}+\|(1-\chi(\varepsilon\cdot))\,A_\eta\|_{\dot H^\sigma}.\end{aligned}$$

By vertue of Lemma 5.3, we have

$$\begin{aligned}\|a_\varepsilon-a\|_{\dot H^\sigma} &\leqslant \|1-\chi(\varepsilon\cdot)\|_{\dot H^{\frac32}\cap L^\infty}\,\|a-A_\eta\|_{\dot H^\sigma}+\|(1-\chi(\varepsilon\cdot))\,A_\eta\|_{\dot H^\sigma}\\ &\leqslant \Big(1+\|\chi\|_{\dot H^{\frac32}\cap L^\infty}\Big)\,\|a-A_\eta\|_{\dot H^\sigma}+\|(1-\chi(\varepsilon\cdot))\,A_\eta\|_{\dot H^\sigma}.\end{aligned}$$

Now, we have just to prove that $\lim\limits_{\varepsilon\to0}\|(1-\chi(\varepsilon\cdot))\,A_\eta\|_{\dot H^\sigma}=0$. This comes from an interpolation argument. For any $0<\sigma<\frac32$ and $\sigma<s<\dfrac32$,

$$\begin{aligned}\|(1-\chi(\varepsilon\cdot))\,A_\eta\|_{\dot H^\sigma} &\leqslant \|(1-\chi(\varepsilon\cdot))\,A_\eta\|_{L^2}^{1-\frac\sigma s}\,\|(1-\chi(\varepsilon\cdot))\,A_\eta\|_{\dot H^s}^{\frac\sigma s}\\ &\leqslant \|(1-\chi(\varepsilon\cdot))\,A_\eta\|_{L^2}^{1-\frac\sigma s}\,\Big(1+\|\chi\|_{\dot H^{\frac32}\cap L^\infty}\Big)\,\|A_\eta\|_{\dot H^s}^{\frac\sigma s}.\end{aligned}$$

To conclude, we have just to notice that the term $\|(1-\chi(\varepsilon\cdot))\,A_\eta\|_{L^2}^{1-\frac\sigma s}$ tends to 0 for ε small enough, by vertue of Lebesgue Theorem. The other term is obviously bounded, since A_η belongs to any Sobolev spaces, for any $\varepsilon>0$, thanks to truncature process. □

6. Appendix B.

In this appendix, we prove a general Theorem about an estimate in the X^s_T-space of a solution of a perturbed Navier-Stokes system. The method is standard: the first step consists in establishing an $\dot H^s$-energy estimate. Then, some computations on scalar-product terms lead to an inequality on which we can apply Gronwall's lemma. In particular, we apply this Theorem to prove that the map $u_0\mapsto T_*(u_0)$ is a lower semi-continous function on $\dot H^s$.

Theorem 6.1. *Let q be an element belonging to the space X^s_T, defined by for any $T<\widetilde T(q)=\widetilde T$*

$$\|q\|^2_{X^s_T}\stackrel{\text{def}}{=}\|q\|^2_{L^\infty_T(\dot H^s)}+\|q\|^2_{L^2_T(\dot H^{s+1})}.$$

Let r be a solution of the following perturbed Navier-Stokes system

$$\left\{\begin{array}{rcl} \partial_t r + r.\nabla r - \Delta r + r\cdot\nabla q + q.\nabla r & = & -f - \nabla p \\ \operatorname{div} r & = & 0 \\ r_{|t=0} & = & r_0. \end{array}\right.$$

Let $\varepsilon_0 > 0$. Let T_0 be the time defined by

$$T_0 \stackrel{\text{def}}{=} \sup\big\{0 < T < \widetilde{T}(q) \mid \|r(t)\|^2_{L^\infty_T(\dot{H}^s)} \leqslant \varepsilon_0\big\}.$$

Then, for any $t \leqslant T_0$, we have

$$\|r\|^2_{X^s_T} \lesssim \Big(\|r_0\|^2_{\dot{H}^s} + \|f\|^2_{L^2_T(\dot{H}^{s-1})}\Big) \exp\Big(\varepsilon_0^{\frac{2}{2s-1}}\, \widetilde{T} + \widetilde{T}^{s-\frac12}\, \|q\|^2_{X^s_T} + \widetilde{T}\, \|q\|^{\frac{4}{2s-1}}_{L^\infty_T(\dot{H}^s)}\Big).$$

Proof. A $\dot{H}^s$ scalar-product, time integration and triangular inequality yield

$$\begin{aligned} \|r\|^2_{\tilde{X}^s_T} &\stackrel{\text{def}}{=} \|r\|^2_{\dot{H}^s} + 2\int_0^t \|r(t')\|^2_{\dot{H}^{s+1}} dt' \\ &\leqslant \|r_0\|^2_{\dot{H}^s} + 2\int_0^t \big|\big((r\cdot\nabla r)\mid r\big)_{\dot{H}^s}\big|\, dt' + 2\int_0^t \big|\big((q\cdot\nabla r)\,|\, r\big)_{\dot{H}^s}\big|\, dt' \\ &\quad + 2\int_0^t \big|\big((r\cdot\nabla q)\,|\, r\big)_{\dot{H}^s}\big|\, dt' + 2\int_0^t \big|\big(f|\, r\big)_{\dot{H}^s}\big|\, dt'. \end{aligned} \tag{66}$$

We assess each term in the right-hand side; the divergence-free condition implies

$$\begin{aligned} \big|\big((r\cdot\nabla r)\mid r\big)_{\dot{H}^s}\big| &\leqslant \|r\cdot\nabla r\|_{\dot{H}^{s-1}}\, \|r\|_{\dot{H}^{s+1}} \\ &\leqslant \|r\otimes r\|_{\dot{H}^s}\, \|r\|_{\dot{H}^{s+1}}. \end{aligned}$$

Thanks to Corollary 5.4, we infer that

$$\big|\big((r\cdot\nabla r)\mid r\big)_{\dot{H}^s}\big| \leqslant C(s)\, \|r\|^{s+\frac12}_{\dot{H}^s}\, \|r\|^{\frac52-s}_{\dot{H}^{s+1}}.$$

Then, integrating in time and applying Young's inequality $\big(ab \leqslant \dfrac{a^p}{p} + \dfrac{b^{p'}}{p'},\ \text{with } \dfrac{1}{p} + \dfrac{1}{p'} = 1\big)$ yield

$$\begin{aligned} \int_0^t \big|\big((r\cdot\nabla r)\,|\, r\big)_{\dot{H}^s}\big|\, dt' &\leqslant C(s) \int_0^t \|r\|^{s+\frac12}_{\dot{H}^s}\, \|r\|^{\frac52-s}_{\dot{H}^{s+1}} dt' \\ &\leqslant C(s) \int_0^t \|r\|^{2\,\frac{2s+1}{2s-1}}_{\dot{H}^s}\, dt' + \frac{1}{12}\int_0^t \|r\|^2_{\dot{H}^{s+1}} dt'. \end{aligned} \tag{67}$$

Now we have to estimate $\displaystyle\int_0^t \big|\big((r\cdot\nabla q)\,|\, r\big)_{\dot{H}^s}\big|\, dt'$ and $\displaystyle\int_0^t \big|\big((q\cdot\nabla r)\,|\, r\big)_{\dot{H}^s}\big|\, dt'$. Actually, thanks to the divergence-free condition, it is exactly the same estimate and we get

$$\begin{aligned} \big|\big((r\cdot\nabla q)\,|\, r\big)_{\dot{H}^s}\big| &\leqslant \|r\cdot\nabla q\|_{\dot{H}^{s-1}}\, \|r\|_{\dot{H}^{s+1}} \\ &\leqslant \|r\otimes q\|_{\dot{H}^s}\, \|r\|_{\dot{H}^{s+1}}. \end{aligned}$$

Once again, Corollary 5.4 gives

$$\begin{aligned} \int_0^t \big|\big((r\cdot\nabla q)\,|\, r\big)_{\dot{H}^s}\big|\, dt' &\leqslant C(s) \int_0^t \|r\|_{\dot{H}^s} \|q\|^{s-\frac12}_{\dot{H}^s}\, \|q\|^{\frac32-s}_{\dot{H}^{s+1}}\, \|r\|_{\dot{H}^{s+1}} dt' \\ &\quad + C(s) \int_0^t \|q\|_{\dot{H}^s}\, \|r\|^{s-\frac12}_{\dot{H}^s}\, \|r\|^{\frac52-s}_{\dot{H}^{s+1}} dt'. \end{aligned}$$

Young's inequality implies

$$
\begin{aligned}
\int_0^t \big|\big((r\cdot\nabla q)\,|\, r\big)_{\dot H^s}\big|\, dt' &\leqslant C(s)\int_0^t \|r\|^2_{\dot H^s}\|q\|^{2s-1}_{\dot H^s}\,\|q\|^{3-2s}_{\dot H^{s+1}}\,dt' \\
&\quad + C(s)\int_0^t \|q\|^{\frac{4}{2s-1}}_{\dot H^s}\,\|r\|^2_{\dot H^s}dt' + \frac{2}{12}\int_0^t \|r\|^2_{\dot H^{s+1}}dt'.
\end{aligned}
\tag{68}
$$

Same arguments give an estimate of exterior force term

$$
\begin{aligned}
\int_0^t \big|(f|\,r)_{\dot H^s}\big|dt' &\leqslant \int_0^t \|f\|_{\dot H^{s-1}}\,\|r\|_{\dot H^{s+1}}\,dt' \\
&\leqslant C\int_0^t \|f\|^2_{\dot H^{s-1}}dt' + \frac{1}{12}\int_0^t \|r\|^2_{\dot H^{s+1}}dt'.
\end{aligned}
\tag{69}
$$

Combining Inequalities (66), (67), (68) and (69), we get

$$
\begin{aligned}
\|r\|^2_{\tilde X^s_T} &\leqslant \|r_0\|^2_{\dot H^s} + C\int_0^t \|f\|^2_{\dot H^{s-1}}dt' + 2\int_0^t \frac{6}{12}\|r\|^2_{\dot H^{s+1}}dt' \\
&\quad + C(s)\int_0^t \|r\|^2_{\dot H^s}\Big(\|r\|^{\frac{4}{2s-1}}_{\dot H^s} + \|q\|^{2s-1}_{\dot H^s}\,\|q\|^{3-2s}_{\dot H^{s+1}} + \|q\|^{\frac{4}{2s-1}}_{\dot H^s}\Big)dt'.
\end{aligned}
\tag{70}
$$

Let us introduce the time T_0 defined by

$$
T_0 \stackrel{\text{def}}{=} \sup\big\{0<T<\widetilde T \mid \|r(t)\|^2_{L^\infty_T(\dot H^s)} \leqslant \varepsilon_0\big\}.
$$

Therefore, for any $t\leqslant T_0$, we have

$$
\begin{aligned}
\|r\|^2_{X^s_T} &\stackrel{\text{def}}{=} \|r\|^2_{\dot H^s} + \int_0^t \|r(t')\|^2_{\dot H^{s+1}}dt' \\
&\lesssim \|r_0\|^2_{\dot H^s} + \int_0^t \|f\|^2_{\dot H^{s-1}}dt' + \int_0^t \|r\|^2_{\dot H^s}\Big(\varepsilon_0^{\frac{2}{2s-1}} + \|q\|^{2s-1}_{\dot H^s}\,\|q\|^{3-2s}_{\dot H^{s+1}} + \|q\|^{\frac{4}{2s-1}}_{\dot H^s}\Big).
\end{aligned}
$$

Thanks to Gronwall's lemma, we infer that for any $T<T_0\leqslant\widetilde T$

$$
\|r\|^2_{X^s_T} \lesssim \Big(\|r_0\|^2_{\dot H^s} + \|f\|^2_{L^2_T(\dot H^{s-1})}\Big)\,\exp\Big(\varepsilon_0^{\frac{2}{2s-1}}\,\widetilde T + \widetilde T^{s-\frac12}\,\|q\|^2_{X^s_T} + \widetilde T\,\|q\|^{\frac{4}{2s-1}}_{L^\infty_T(\dot H^s)}\Big).
$$

This concludes the proof Theorem 6.1. □

Proposition below is well-known and can be seen as a consequence of Theorem 6.1. We perturb a data by a small term and we are interesting in the consequence on the lifespan of the Navier-Stokes solution associated with such a perturbed data. The lifespan of perturbed Navier-Stokes solution can not decrease too much, compared to the lifespan of the non-perturbed one. More precisely, we have the following proposition.

Proposition 6.2. *The map $u_0\mapsto T_*(u_0)$ is a lower semi-continous function on $\dot H^s$*

$$
\textit{e.g.}\quad \forall\varepsilon>0,\exists\alpha>0,\ \forall v_0 \textit{ in } \dot H^s\ \textit{ such that } \|v_0\|_{\dot H^s}<\alpha,\ \textit{then } T_*(u_0+v_0)\geqslant T_*(u_0)-\varepsilon.
$$

Moreover, (under notations of Theorem 6.1), a constant $C>0$ exists such that for any $T\leqslant T_*(u_0)-\varepsilon$

$$
\begin{aligned}
\|NS(u_0+v_0)-NS(u_0)\|^2_{X^s_T} &\leqslant C\,\|v_0\|^2_{\dot H^s} \\
&\quad\times\ \exp\Big(\varepsilon_0^{\frac{2}{2s-1}}\,T + T^{s-\frac12}\,\|NS(u_0)\|^2_{X^s_T} + T\,\|NS(u_0)\|^{\frac{4}{2s-1}}_{L^\infty_T(\dot H^s)}\Big).
\end{aligned}
\tag{71}
$$

Proof. Let u_0 and v_0 two elements in $\dot H^s$. We operate a small perturbation of the data u_0 by v_0 (the aim is to quantify this smallness condition) and we want to prove that the lifespan of the perturbed Navier-Stokes solution $NS(u_0+v_0)$ can not be much less than the lifespan of $NS(u_0)$. The process is standard. We introduce an error term R defined by

$$
R(t,x) = NS(u_0+v_0) - NS(u_0).
$$

Classical computations imply that R is solution of the following perturbed Navier-Stokes system

$$\left\{\begin{array}{rcl}\partial_t R + R\cdot\nabla R - \Delta R + R\cdot\nabla NS(u_0) + NS(u_0)\cdot\nabla R & = & -\nabla p\\ \operatorname{div} R & = & 0\\ R_{|t=0} & = & v_0.\end{array}\right. \tag{72}$$

Let $\varepsilon_0 > 0$. Let us introduce the time T_0 defined by

$$T_0 := \sup\big\{0 < T < T_*(u_0) \mid \|R(t)\|^2_{L^\infty_T(\dot H^s)} \leqslant \varepsilon_0\big\}.$$

Thanks to Theorem 6.1, we infer that for any $T \leqslant T_0$

$$\|R\|^2_{X^s_T} \leqslant C\,\|v_0\|^2_{\dot H^s}\,\exp\Big(\varepsilon_0^{\frac{2}{2s-1}}\,T + T^{s-\frac12}\,\|NS(u_0)\|^2_{X^s_T} + T\,\|NS(u_0)\|^{\frac{4}{2s-1}}_{L^\infty_T(\dot H^s)}\Big). \tag{73}$$

The above expression gives the smallness condition on $\|v_0\|_{\dot H^s}$. Indeed, suppose that v_0 satifies

$$C\,\|v_0\|^2_{\dot H^s}\,\exp\Big(\varepsilon_0^{\frac{2}{2s-1}}\,T + T^{s-\frac12}\,\|NS(u_0)\|^2_{X^s_T} + T\,\|NS(u_0)\|^{\frac{4}{2s-1}}_{L^\infty_T(\dot H^s)}\Big) \leqslant \varepsilon_0. \tag{74}$$

Therefore, the error term R, keeps on living until the time $T_*(u_0) - \varepsilon$, for any $\varepsilon > 0$. This concludes the proof of the proposition. $\square$

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